Martingale Optimal transport

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Outline

Analytic formulation (Monge 1781)

- Initial distribution : probability measure μ
- Final distribution : probability measure ν

Problem : find an optimal transference plan T^*

$$P_2^M := \sup_{T \in \mathcal{T}(\mu,\nu)} \int c(x, T(x))\mu(dx)$$

where $\mathcal{T}(\mu,\nu)$ of all maps $T : x \longmapsto y = T(x)$ such that

$$\nu = \mu \circ T^{-1}$$

Probabilistic formulation (Kantorovich 1942)

Randomization of transference plans :

$$\overline{P}_2^K := \sup_{\mathbb{P} \in \mathcal{P}_2(\mu,\nu)} \int c(x,y)\mathbb{P}(dx,dy)$$

where $\mathcal{P}_2(\mu,\nu)$ is the collection of all joint probability measures with marginals μ and ν

Example : $c(x,y) = -|x-y|^2 \Longrightarrow$ maximization of correlations :

$$\sup_{\mathbb{P} \in \mathcal{P}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[XY]$$

Kantorovich duality

Duality in linear programming, Legendre-Fenchel duality...

$$D_2^0 := \inf_{(\varphi,\psi)\in\mathcal{D}_2^0} \int \varphi d\mu + \int \psi d\nu$$

$$\mathcal{D}_2^0 := \left\{(\varphi,\psi) : \varphi^+ \in \mathbb{L}^1(\mu), \psi^+ \in \mathbb{L}^1(\nu), \varphi \oplus \psi \geq c\right\}$$

where $\varphi \oplus \psi(x,y) := \varphi(x) + \psi(y)$

- Inequality $D_2^0 \geq P_2^K$ obvious
- Reverse inequality needs Hahn-Banach theorem

One-dimensional Version of the Brenier Theorem

Rachev and Rüschendorf

Back to the original Monge formulation

- $P_2^K \geq P_2^M$: Kantorovitch formulation $\equiv$ relaxation of Monge one

Theorem (Y. Brenier)

Let $c \in C^1$ with $c_{xy} > 0$. Assume μ has no atoms. Then there is a unique optimal transference plan :

$$\mathbb{P}^*(dx, dy) = \mu(dx)\delta_{\{T^*(x)\}}(dy) \quad with \quad T^* = F_\nu^{-1} \circ F_\mu$$

Consequently $P_2^M = P_2^K$, and T^* solves both problems.

- T^* : monotone rearrangement, Frechet-Hoeffding coupling
- $c_{xy} > 0$: Spence-Mirrlees condition

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On the Spence Mirrlees condition

The solution of the Kantorovitch optimal transportation problem

$$\overline{P}_2^K \quad := \quad \sup_{\mathbb{P}\in\mathcal{P}_2(\mu,\nu)} \int c(x,y)\mathbb{P}(dx,dy)$$

is not modified by the change of performance criterion :

$$c(x,y) \quad \longrightarrow \quad \hat{c}(x,y) := c(x,y) + a(x) + b(y)$$

Notice that the Spence Mirrlees condition $c_{xy} > 0$ is stable by this transformation

Lower bound

Set $\bar{c}(\bar{x}, y) := -c(-\bar{x}, y)$. Then

$$\inf_{\mathbb{P}\in\mathcal{P}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c(X,Y)] = - \sup_{\mathbb{P}\in\mathcal{P}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[\bar{c}(-\bar{X},Y)]$$

where

- $\bar{X} := -X \sim \bar{\mu}$ with c.d.f. $F_{\bar{\mu}}(\bar{x}) := 1 - F_{\mu}(-\bar{x})$

- $\bar{c}$ satisfies the Spence Mirrlees condition, whenever c does. So, the lower bound is attained by the anti-monotone transference plan :

$$\mathbb{P}_*(dx, dy) := \mu(dx)\delta_{\{T_*(x)\}}(dy), \qquad T_*(x) := F_\nu^{-1} \circ F_{\bar{\mu}}$$

Financial Interpretation

Financial interpretation

- $X \sim \mu$ and $Y \sim \nu$ prices of two assets at time 1

- μ and ν identified from market prices of call options :

$$C_\mu(K) = \int (x-K)^+ \mu(dx), \qquad C_\nu(K) = \int (y-K)^+ \nu(dy)$$

(Breeden-Litzenberger 1978)

- $c(X, Y)$ payoff of derivative security

- Robust bounds on dervative's price :

$$\inf_{\mathbb{P} \in \mathcal{P}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c(X,Y)] \quad \text{and} \quad \sup_{\mathbb{P} \in \mathcal{P}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c(X,Y)]$$

Financial interpretation of the dual problem

- $\varphi(X), \psi(Y)$: optimal Vanilla position in Assets X and Y

- Can be expressed as a combination of calls/puts (Carr-Madan) :

$$g(s) = g(s^*)+(s-s^*)g'(s^*)+\int_0^{s^*}(K-s)^+g''(K)dK+\int_{s^*}^{\infty}(s-K)^+g''(K)dK$$

so their market market prices are $\int \varphi d\mu$ and $\int \psi d\nu$

- With $\mathcal{D}_2^0 := \{(\varphi,\psi) : \varphi^+ \in \mathbb{L}^1(\mu), \psi^+ \in \mathbb{L}^1(\nu), \varphi \oplus \psi \geq c\}$:

$$D_2^0 \;=\; \inf_{(\varphi,\psi)\in\mathcal{D}_2^0} \int \varphi(x)\mu(dx) + \int \psi(y)\nu(dy)$$

is the cheapest static position in X and Y so as to superhedge $c(X, Y)$

Martingale Optimal Transport

One asset observed at two future dates

Our interest now is on the case where

$$X = X_0 \quad \text{and} \quad Y = X_1$$

are the prices of the same asset at two future dates 0 and 1

Interest rate is reduced to zero

This setting introduces a new feature :

- the possibility of dynamic trading the asset between times 0 and 1
- duality converts this possibility into the martingale condition $\mathbb{E}^{\mathbb{P}}[Y|X] = X$

Superhedging problem $\equiv$ Kantorovitch dual

Robust super hedging problem naturally formulated as :

$$v_0 \;=\; D_2(\mu,\nu) \;=\; \inf_{(\varphi,\psi,h)\in\mathcal{D}_2} \left\{\mu(\varphi)+\nu(\psi)\right\}$$

where $\mu(\varphi)=\int\varphi d\mu$, $\mu(\psi)=\int\psi d\nu$, and

$$\mathcal{D}_2 := \Big\{(\varphi,\psi,h) \;:\; \varphi^+\in\mathbb{L}^1(\mu), \psi^+\in\mathbb{L}^1(\nu), h\in\mathbb{L}^0 \quad \varphi\oplus\psi+h^{\otimes}\geq c\Big\}$$

$\varphi\oplus\psi(x,y) := \varphi(x)+\psi(y)$ and $h^{\otimes}(x,y) := h(x)(y-x)$

The Martingale Optimal Transportation Problem

The corresponding dual problem is :

$$P_2(\mu,\nu) \quad := \quad \sup_{\mathbb{P}\in\mathcal{M}_2(\mu,\nu)} \quad \mathbb{E}^{\mathbb{P}}\big[c(X,Y)\big]$$

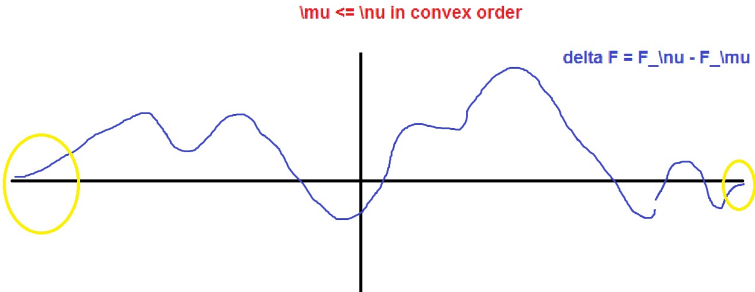
where $\mathcal{M}_2(\mu,\nu) := \big\{\mathbb{P}\in\mathcal{P}_2(\mu,\nu) : \mathbb{E}^{\mathbb{P}}[Y|X] = X\big\}$

and we recall $\mathcal{P}_2(\mu,\nu) := \big\{\mathbb{P}\in\mathcal{P}_{\mathbb{R}^2} : X\sim_{\mathbb{P}}\mu,\, Y\sim_{\mathbb{P}}\nu\big\}$

Implication of the convex ordering

Strassen 1965 : $\mathcal{M}_2(\mu,\nu) \neq \emptyset$ iff μ and ν have same mean and $\mu \preceq \nu$ (convex), i.e. with $\delta F := F_\nu - F_\mu$

$$\int \delta F(\xi)d\xi = 0 \quad \text{and for all } k \quad \int_{(-\infty,k)} \delta F(\xi)d\xi \geq 0$$



Outline

Worst Case Financial Market – Brenier Theorem

- The solution $\mathbb{P}^* \in \mathcal{M}_2(\mu, \nu)$ always exists

- **Question 1** : Is there an optimal transfert map, i.e. optimal transport of μ to ν through a map T^* ? (Brenier Theorem)

Can not be a map, unless $\mu = \nu$!

- **Question 2** : Is there a transference plan along a minimal randomization

X → $Y = T_u(X)$ with probability $q(X)$

X → $Y = T_d(X)$ with probability $1 - q(X)$

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Previous literature : Beiglbock and Juillet (2012)

Definition

$\mathbb{P} \in \mathcal{M}_2(\mu, \nu)$ is **left-monotone** if $\mathbb{P}[(X, Y) \in \Gamma] = 1$, for some $\Gamma \subset \mathbb{R} \times \mathbb{R}$, and

$$\text{for all } (x, y_1), (x, y_2), (x', y') \in \Gamma : \quad x < x' \implies y' \notin (y_1, y_2)$$

Theorem

- *There exists a left-monotone martingale transport*
- *Assume μ has no atoms. Then, any left-monotone $\mathbb{P} \in \mathcal{M}_2(\mu, \nu)$ is concentrated on two graphs*

$$\mathbb{P} = \mu(dx)\left[q(x)\delta_{\{T_u(x)\}}(dy)(1-q)(x)\delta_{\{T_d(x)\}}(dy)\right]$$

Previous literature : Beiglbock and Juillet (2012)

Theorem

$\mu_2 \succeq \mu_1$, μ_1 *without atoms. Then :*
(i) there exists a unique left-monotone transport plan $\mathbb{P}^*$
(ii) $\mathbb{P}^*$ *is a solution* $P_2(\mu, \nu)$ *in the following cases :*

- $c(x, y) = h(x - y)$ *with* h' *strictly convex,*
- $c(x, y) = \varphi(x)\psi(y)$, $\varphi, \psi \geq 0$, ψ *strict convex,* φ *decreasing*

Our objective :

- explicit derivation of $\mathbb{P}^*$
- extend the class of couplings c for which $\mathbb{P}^*$ is optimal
- extend to the multi-marginals case

Explicit left-monotone transference plan

Theorem

Let μ, ν have finite first moment, same mean, $\mu \preceq \nu$, and μ without atoms. Then, the unique left-monotone transference plan is

$$\mathbb{P}^*(dx, dy) = \big[q(x)\delta_{T_d(x)}(dx) + (1-q)(x)\delta_{T_u(x)}(dx)\big]\mu(dx)$$

where T_u, T_d are explicitly defined as follows...
In particular, outside jumps, T_u and T_d solve the following ODEs :

$$d(\delta F \circ T_d) = (1-q)dF_\mu, \quad d(F_\nu \circ T_u) = qdF_\mu$$

Duality and explicit Martingale Version of the Brenier Theorem

Theorem

Let μ, ν have finite first moment, same mean, $\mu \preceq \nu$, and μ without atoms. Assume that $c_{xyy} > 0$. Then

$$P_2 = D_2$$

and there is an explicit dual optimizer (φ^*, ψ^*, h^*) defined as follows...

The martingale version of the Spence-Mirrlees condition

... is $c_{xyy} > 0$:

• Notice that the solution of the Martingale Transport problem is not altered by the change of performance criterion :

$$c(x,y) \quad \longrightarrow \quad \hat{c}(x,y) := c(x,y) + a(x) + b(y) + h(x)(y-x)$$

• $\hat{c}_{xyy} = c_{xyy}$

• The conditions of Beiglbock and Juillet :

- $c(x,y) = h(x-y)$ with h' strictly convex,
- $c(x,y) = \varphi(x)\psi(y)$, $\varphi, \psi \geq 0$, ψ strict convex, φ decreasing

satisfy $c_{xyy} > 0$

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Lower bound

Suppose $c_{xyy} > 0$. Then

$$\bar{c}(\bar{x}, \bar{y}) := -c(-\bar{x}, -\bar{y}) \quad \text{satisfies} \quad \bar{c}_{\bar{x}\bar{y}\bar{y}} > 0$$

We exploit this symmetry to derive the lower bound :

$$\begin{aligned} \inf_{\mathbb{P}\in\mathcal{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[c(X,Y)\big] &= -\sup_{\mathbb{P}\in\mathcal{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[\bar{c}(\bar{X},\bar{Y})\big] \\ &= \mathbb{E}^{\mathbb{P}_*}\big[c(X,Y)\big] \end{aligned}$$

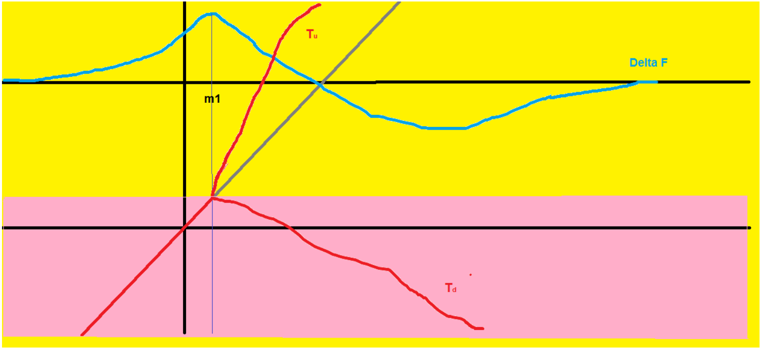
where $\mathbb{P}_*$ is the left-monotone transference plan constructed from

$$F_{\bar{\mu}}(\bar{x}) := 1 - F_{\mu}(-\bar{x}) \quad \text{and} \quad F_{\bar{\nu}}(\bar{y}) := 1 - F_{\nu}(-\bar{y}-)$$

Construction : One local maximizer of δF

Easy case : $T_u \nearrow$ and $T_d \searrow$ after m_1, and

$$\mathbb{P}^*(dx, dy) \quad = \quad \mu_0(dx)\Big[q(x)\delta_{\{T_u(x)\}}(dy) + (1 - q(x))\delta_{\{T_d(x)\}}(dy)\Big]$$



Martingale transportation constraints

- First marginal is μ_0, Martingale condition holds if $q \in [0,1]$
- Second marginal :
  - either $y \le m_1$, then $\mathbb{P}_*[Y \in dy] = dF_\mu(y) + \mathbb{E}\big[(1-q)(X)\mathbb{1}_{\{T_d(X)\in dy\}}\big]$. So $Y \sim_{\mathbb{P}_*} \nu$ with decreasing T_d implies

$$d(\delta F \circ T_d) \;=\; -(1-q)dF_\mu,$$

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The Kantorovitch Dual Side

So far, we have :

$$\mathbb{E}^{\mathbb{P}_*}[c(X,Y)] \quad \le \quad \sup_{\mathcal{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c(X,Y)] \le \inf_{\mathcal{D}_2} \left\{\mu(\varphi)+\nu(\psi)\right\}$$

Our next goal is to construct

$$(\varphi_*,\psi_*,h_*)\in\mathcal{D}_2 \quad \text{such that} \quad \mu(\varphi_*)+\nu(\psi_*)=\mathbb{E}^{\mathbb{P}_*}[c(X,Y)]$$

In particular, this would imply duality and existence hold

$\Longrightarrow$ $\varphi_*(X)+\psi_*(Y)+h_*(X)(Y-X)-c(X,Y)=0$, $\mathbb{P}_*-$a.s.

$\Longrightarrow$ $\varphi_*(x)=\max_{y\in\mathbb{R}}\{c(x,y)-\psi_*(y)-h_*(x)(y-x)\}$, $x\in\mathbb{R}$

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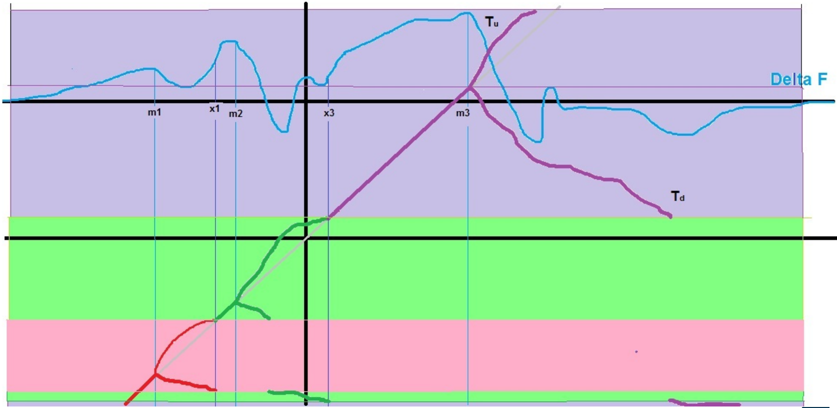
In particular, this would imply duality and existence hold

$$\Longrightarrow \varphi_*(X)+\psi_*(Y)+h_*(X)(Y-X)-c(X,Y)=0,\ \mathbb{P}_*\text{−a.s.}$$

$$\Longrightarrow \varphi_*(x) = \max_{y\in\mathbb{R}}\{c(x,y)-\psi_*(y)-h_*(x)(y-x)\},\ x\in\mathbb{R}$$

Multiple local maxima of δF



Outline

Finitely many marginals martingale transportation

- Extension to finite discrete-time is immediate :
  - μ_i have same mean, and $\mu_n \succeq \ldots \succeq \mu_0$
  - Optimal transportation with n marginals constraint :

$$P_n(\mu) = \sup_{\mathbb{P}\in\mathcal{M}_n(\mu)} \mathbb{E}^{\mathbb{P}}[c(X)], \qquad c(x_1,\ldots,x_n) = \sum_{i=1}^{n-1} c^i(x_i, x_{i+1})$$

  - The dual problem :

$$D_n(\mu) \;:=\; \inf_{(u,h)\in\mathcal{D}_n} \sum_{i=1}^{n} \mu_i(u_i),$$

where

$$\mathcal{D}_n := \Big\{(u,h) : (u_i)^+ \in \mathbb{L}^1(\mu_i) \text{ and } \oplus_{i=1}^n u_i + \textstyle\sum_{i=1}^{n-1} h_i^{\otimes^i} \geq c\Big\}.$$

Martingale Transportation under finitely many marginals constraints

Theorem

Suppose $\mu_1 \preceq \ldots \preceq \mu_n$ in convex order, with finite first moment, same mean, and $\mu_1, \ldots, \mu_{n-1}$ have no atoms. Assume further that $c^i_{xyy} > 0$. Then, the strong duality holds, the transference plan

$$\mathbb{P}^*_n(dx) = \mu_1(dx_1) \prod_{i=1}^{n-1} T^i_*(x_i, dx_{i+1})$$

is optimal for the martingale transportation problem $P_n(\mu)$, and (u^*, h^*) is optimal for the dual problem $D_n(\mu)$

Example : applies to the discrete monitoring variance swap : $c(x_1, \ldots, x_n) := \sum_{i=1}^{n} \left(\ln \frac{x_i}{x_{i-1}}\right)^2$

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Example : applies to the discrete monitoring variance swap :
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Continuous-Time Limit

One maximizer m of $F_{\mu_1} - F_{\mu_0}$: first asymptotics

Suppose $F_{\mu_1}(x) = F_{\mu_0}(x) + \varepsilon\delta(x) + \circ(\varepsilon)$

Lemma

$T_u^\varepsilon(x) = x + \varepsilon j_u(x) + \circ(\varepsilon)$ *and* $T_d^\varepsilon(x) = x - j_d(x) + O(\varepsilon)$, *where*

$$j_u(x) := \frac{\delta(x - j_d(x)) - \delta(x)}{f_{\mu_0}(x)} \quad \text{and} \quad \int_{x-j_d(x)}^{x} (x-\xi)\delta(\xi)d\xi = 0$$

Assumptions

Assumption $(\mu_t)_{t\in[0,1]}$ have finite first moment, nondecreasing in convex order, with smooth cdf $F(t,x)$, and

- $x \longmapsto \partial_t F(t,x)$ has a unique C^0 maximizer $m(t)$
- $x \longmapsto F(t+h,x) - F(t,x)$ has a unique maximizer $m^h(t)$, $m^h \longrightarrow m$, uniformly
- $f(t,x) := \partial_x F(t,x) > 0$ on its support (ℓ_t, r_t)

The continuous-time dynamics

- $\pi^n : 0 = t_0^n < \cdots < t_n^n = 1$ with $|\pi^n| := \max_i |t_i^n - t_{i-1}^n| \longrightarrow 0$
- $X^n := \big(X^n_{t_i^n}\big)_{0 \le i \le n}$ discrete time Markov martingale $\sim \mathbb{P}_n^*$

Theorem

$X^n \longrightarrow X^*$, weakly. X^* is a pure (downward) jump martingale :

$$dX_t^* = \mathbb{1}_{\{X_{t-} > m(t)\}} j_d(t, X_{t-})(dN_t - \nu_t dt),$$

$\nu_t := \frac{j_u}{j_d}(t, X_{t-})\mathbb{1}_{\{X_{t-} > m(t)\}}$, *and N is a pure jump process with predictable compensator ν. Moreover :*

$$X_t^* \sim \mu_t \quad \text{for all} \quad t \in [0, 1]$$

Examples of Peacocks

X^* is a Peacock (PCOC) in the terminology of Yor

• Fake Brownian motion : $\mu_t = \mathcal{N}(0,t)$, $m(t) = -\sqrt{t}$

• self-similar martingales : $\{M_{c^2 t}, t \geq 0\} \sim \{cM_t, t \geq 0\}$...

Madan and Yor (2002)
Hamza and Klebaner (2007)
Oleszkiewicz (2008)
Hirsch, Profeta, Roynette, Yor (2011)

Model-free super hedging strategy in continuous-time

Theorem

Let $c(x,x) = c_y(x,x) = 0$, and $c_{xyy} > 0$. Then, there exist explicit functions $h^*(t,x)$, $\phi_0^*(x)$, $\phi_1^*(x)$, and $\phi^*(t,x)$ such that

$$\phi_0^*(X_0) + \phi_1^*(X_1) + \int_0^1 \phi^*(t,X_t)dt + \int_0^1 h^*(t,X_t)dX_t$$
$$\geq \xi(X_.) := \frac{1}{2}\int_0^1 c_{yy}(X_t,X_t)d[X^c]_t + \sum_{0<t\leq 1} c(X_{t-},X_t)$$

in the sense of

- *quasi-sure stochastic analysis, i.e. $\mathbb{P}-$a.s. for all martingale measure $\mathbb{P}$*
- *pathwise Föllmer Itô calculus (under additional smoothness)*

Cheapest Model-Free Superhedging

Theorem

Under all previous conditions, $P = D$. Moreover

- $\mathbb{P}^*$ *solution of* P
- (ϕ^*, h^*) *explicit solution of* D
- *Cheapest superhedging cost for the path-dependent option* $\xi(X_.)$:

$$P = D = \int_0^1 \frac{\dot{j}_u}{\dot{j}_d}(t,x)c(x, x - j_d(t,x))f(t,x)\,dx\,dt$$

An extremal Peacock

Unlike the examples in the previous literature on Peacocks (Hamza & Klebaner, Oleszkiewicz, Hirsch-Profeta-Roynette-Yor), our Peacock X^* enjoys an optimality property with respect to the criterion defined by c

Results of this type were also obtained by Hobson and Klimmek (2012), and Hobson (2013)