Solution of optimal stopping problems by a modification of the payoff function

Ernst Presman

Central Economics & Mathematics Institute, Russian Academy of Sciences

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1. Introduction

The time may be discrete or continuous.

Homogenous in time strong Markov process (family) $Z = (Z_t)_{t\geq 0}$ in $X \bigcup e$, $(X, \mathcal{B})$ is a measurable space, e is an absorbing state. $z \in X$ is initial point.

We assume that Z is defined on some filtered probabilistic space

$\rho(z) \geq 0$ — killing intensity;

$g(z)$ — payoff function, $g(e) = 0$; $c(z)$ — cost of observations, $c(e) = 0$.

$$V(z,\tau) = E_z \left[g(Z_\tau) - \int\limits_0^\tau c(Z_s) ds \right], \qquad V(z) = \sup_\tau V(z,\tau).$$

$$V(z,\tau) = \tilde{E}_z \left[g(Z_\tau) e^{-\int\limits_0^\tau \rho(Z_u) du} ds - \int\limits_0^\tau c(Z_s) e^{-\int\limits_0^s \rho(Z_u) du} ds \right],$$

In discrete time $\int_0^\tau \longrightarrow \sum_0^{\tau-1}$, $\quad \exp(-\int_0^t \rho(Z_u) du) \longrightarrow \prod_{u=0}^{t-1} (1 - \rho(Z_u))$.

First papers in the beginning of 60-th (Dynkin, Shiryaev). Monographs by Shiryaev (1968), (1978). Dynkin, Yushkevich (1967), (1975). A lot of papers after that.

General theory: Peskir, Shiryaev (2006).

One-dimensional diffusion:

Salminen (1985) (Martin's boundary),

Dayanik, Karatzas (2003) (reduction to the standard Brownian motion),

Bronstein, Hughston, Pistorius and Zervos (2006) (one-dimensional diffusion on half-line with $\rho(z)$ and piecewise-constant nondecreasing payoff function).

Peskir, Shiryaev (2006) (smooth fitting).

In discrete time for finite (in some cases countable) state space

Sonin (1999) (state elimination algorithm).

Irle (2006) (forward algorithm)

Guess or construct? Approximation from above and from below.

$C \in \mathcal{B}$ — subset of X,

$$\tau_C = \inf\{t : t \geq 0, \quad Z_t \notin C\}.$$

$$g_C(z) = V(z, \tau_C) = E_z\left[g(Z_{\tau_C}) - \int_0^{\tau_C} c(Z_s)ds\right], \quad g_C(z) = g(z) \text{ if } z \notin C.$$

Main Lemma. *If* $g_C(z) > g(z)$ *for all* $z \in C$, *then the problem with the payoff function* $g_C(z)$ *has the same value function as the problem with the payoff function* $g(z)$.

Proof. It follows from $g_C(z) > g(z)$ that $V_C(z) \geq V(z)$.
From the other side for any τ we can consider $\tau' = \inf\{t : t \geq \tau, Z_t \notin C\}$.
Then

$$V(z,\tau') = E_z\left[-\int_0^{\tau} c(Z_s)ds + E_{z_\tau}\left[g(Z_{\tau'}) - \int_{\tau}^{\tau'} c(Z_s)ds\right]\right] =$$

$$= E_z\left[-\int_0^{\tau} c(Z_s)ds + g_C(z_\tau)\right] = V_C(z,\tau).$$

Consequently $V(z) \geq V_C(z)$.

We say that a function $f(z)$ is a modification of the payoff function $g(z)$ (or is a modified payoff function) if $f(z) \geq g(z)$ and $f(z) = g_C(z)$ for $C = \{z : f(z) > g(z)\}$.

It follows from the Main Lemma that the optimal stopping problems with the payoff function $g(z)$ and the modified payoff function $f(z)$ have the same value function.

How to find such a set C which is a candidate for modification?

2. Discrete time

Revaluation operator $Tf(z) = -c(z) + E_z f(Z_1)$.

$\tilde{V}^{(0)}(z) = g(z),\ \tilde{V}^{(k+1)}(z) = \max[g(z), T\tilde{V}^{(k)}(z)]$. Then $\tilde{V}^{(k)}(z)] \uparrow V(z)$.

Even in a very simple cases $\tilde{V}^{(k)}(z) \neq V(z)$ for all k.

Sonin (1999) – state elimination algorithm for the case of the finite number of states.

It has no sense to stop on the set $\{z : Tg(z) > g(z)\}$.

Sonin proposed to eliminate this set and to consider new Markov chain which coincides with the old one at the times when the old chain is in the complimentary set.

If X consists of n states then sequentially applying this procedure we can find the value function after not more than $2n-2$ steps.

What to do for arbitrary state space?

What to do in continuous time?

The answer to the first question was given in

[1] Ernst Presman. A new approach to the solution of optimal stopping problem in a discrete time. Stochastics: An International Journal of Probability and Stochastic Processes, Special Issue: Optimal stopping with Applications, 83 (2011), n. 4-6, pp. 467 - 475.

Consider operator $Lf(z) = Tf(z) - f(z)$

Very important property:

Function $g_C(z)$ satisfies for $z \in C$ the equation $Lg_C(z) = 0$.

a) If $C = \{z : \ Lg(z) > 0\}$ is empty, then $V(z) = g(z)$,

b) If $C = \{z : \ Lg(z) > 0\}$ is not empty, then $g_C(z)$ is a modification of the payoff function $g(z)$.

Let $g_0(z) = g(z), \ \ C_1 = \{z : Lg(z) > 0\}$,

$$C_{k+1} = \{z : Tg_{C_k}(z) - g(z) > 0\} = C_k \bigcup \{z : Lg_{C_k}(z) > 0\}$$

and $g_k(z) = g_{C_k}(z), \ k \geq 1$. The respective sequence of the modified payoff functions $g_k(z)$ is nondecreasing and converges to the value function.

3.1. One-dimensional diffusion

[2] Ernst Presman. Solution of the optimal stopping problem of one-dimensional diffusion based on a modification of payoff function. Prokhorov and Contemporary Probability Theory - In Honor of Yuri V. Prokhorov, Springer Proceedings in Mathematics and Statistics, Springer Verlag, 2013, v. 33, 347-380.

At first, Brownian motion on the interval $[a, b]$ with $\rho(z) = 0$, $c(z) = 0$.

a) Absorbtion at a and b. b) Absorbtion at a and reflection at b.

c) Case a) and partial reflection at the finite set A_0.

$\mathbf{P}_z[Z_t > z] \to \dfrac{1 + \alpha(z)}{2}$ as $t \to 0$,

where $-1 < \alpha(z) < 1$, $\alpha(z) \neq 0$ iff $z \neq A_0)$.

$C =]c, d[$ as a candidate for modification. $g_{]c,d[}(z)$.

Operators: $Lf(z) := \frac{1}{2}\frac{d^2}{dz^2}f(z)$;

$L_1 f(z) = (1 + \alpha(z))f'_+(z) - (1 - \alpha(z))f'_-(z)$,

$g_{]c,d[}(z)$ satisfies on $]c, d[$ the equalities: $Lg_{]c,d[}(z) = 0$, $L_1 g_{]c,d[}(z) = 0$.

Let $\mathcal{C}$ be the set of functions $f(z)$ satisfying the following properties.

1) $f(z)$ is bounded; $f''(z)$ exists and is finite and continuous on $]a, b[$ with exception of a finite (possibly empty) set A^1; $f(z)$ and $f'(z)$ have left and right limits at points from $A^0 \bigcup A^1$.

2) $f(a)$ and $f(b)$ are finite.

3) The set of points where $Lf(z) > 0$ is either empty or consists of a finite number of intervals. Denote by A^2 the set of the endpoints of these intervals.

Let $A = A^0 \bigcup A^1 \bigcup A^2 = \{z_1, \ldots, z_k\}$, where $a = z_0 < z_1 < \ldots < z_k < z_{k+1} = b$.

In what follows we assume that either $g(z) \in \mathcal{C}$ or there exists a modification of $g(z)$ which belongs to $\mathcal{C}$.

Theorem 1. *Let* a) *the function* $g(z)$ *be continuous*;
b) *the set where* $Lg(z) > 0$ *be empty* ;
c) $L_1 g(z) \leq 0$ *for all* $z \in A$;
d1) *if* a *is reflecting point then* $g'_{+}(a) \leq 0$;
d2) *if* b *is reflecting point then* $-g'_{-}(b) \leq 0$.

Then $V(z) = g(z)$.

At first we shall modify $g(z) \in \mathcal{C}$ in the neighborhood of points of discontinuity in such a way that modified function will be continuous and in $g(z) \in \mathcal{C}$.

After that on intervals where $Lg(z) < 0$.

Generalized smooth fitting:

$L_1 g(z) \leq 0$, $Lg(z) \leq 0$ for $z \neq z_i$, $\qquad L_1 g(z_i) > 0$

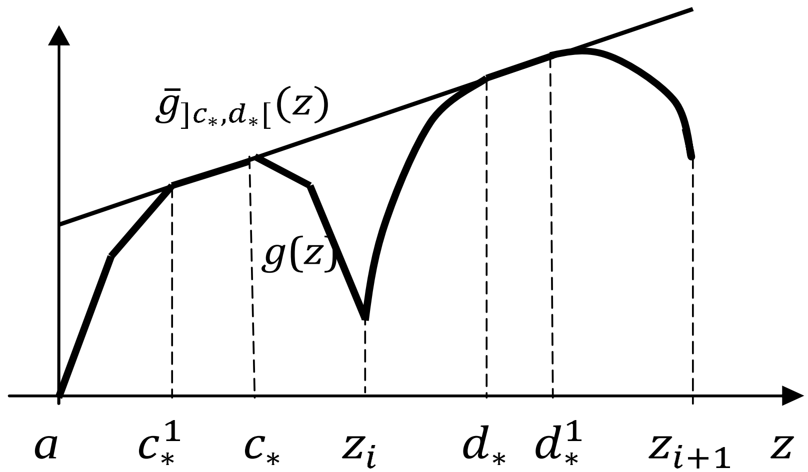

Figure 1:

"Diffusion does not like angles".

Diffusion does not like **convex** angles (i.e. such angles that $L_1 g(z_i) > 0$).

The reason is that in the neighborhood of such angle the payoff function may be modified.

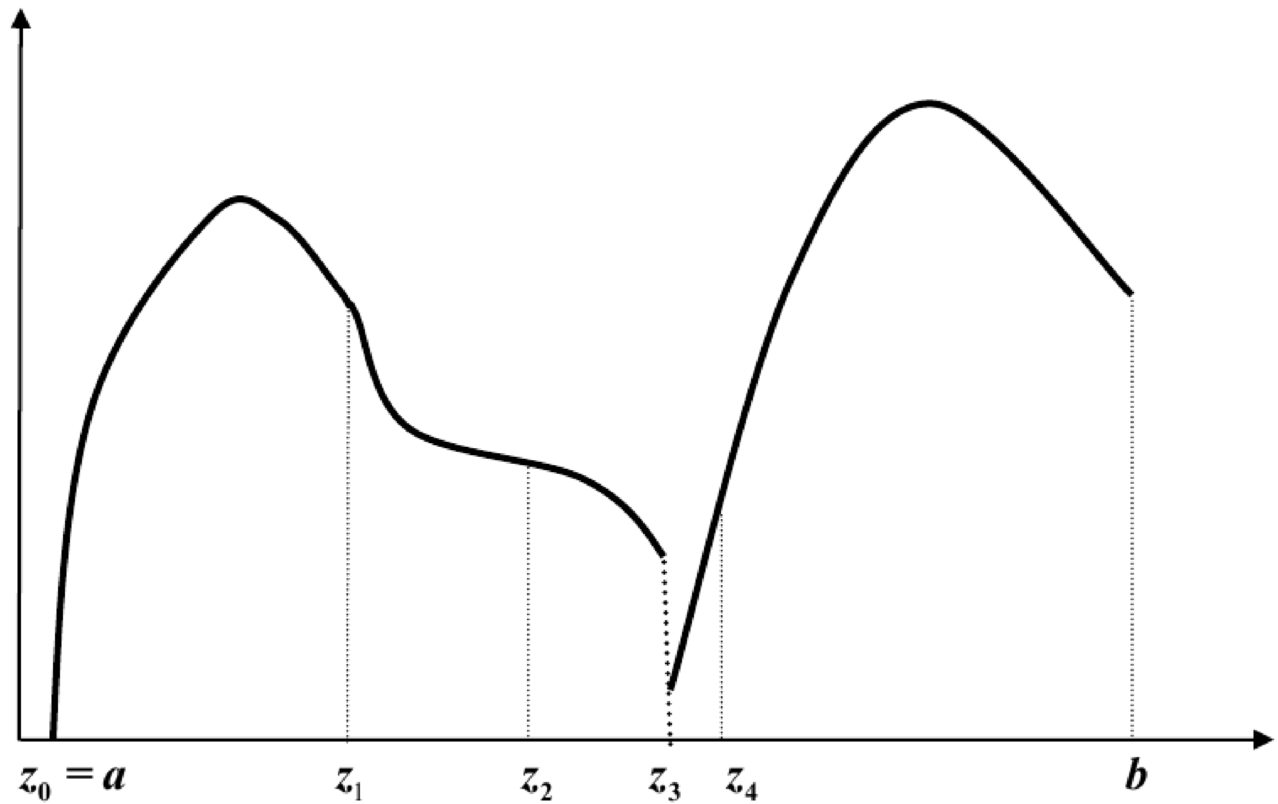

Fig. 1. Payoff function.

Nonstandard points: points of discontinuities, ends of intervals where $Lg(z) > 0$.

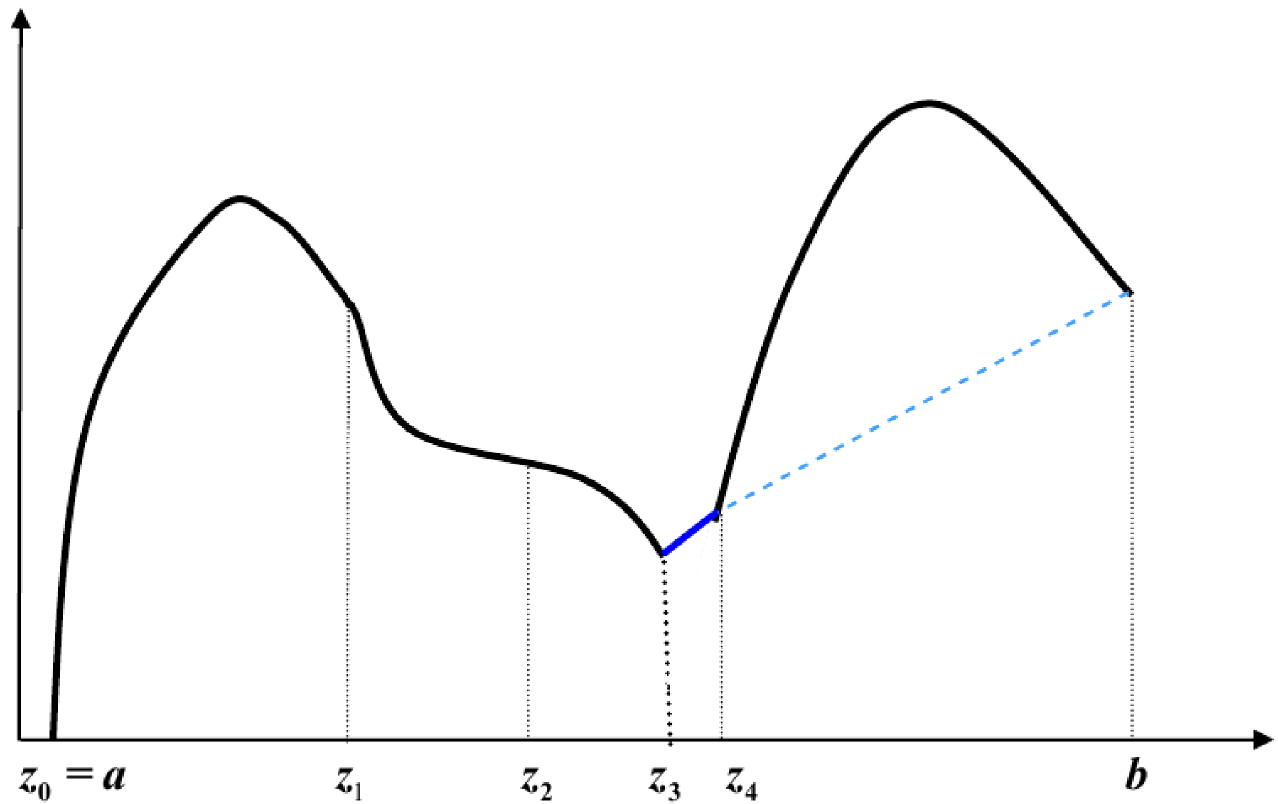

Fig. 1.1 Modification near the points of discontinuities

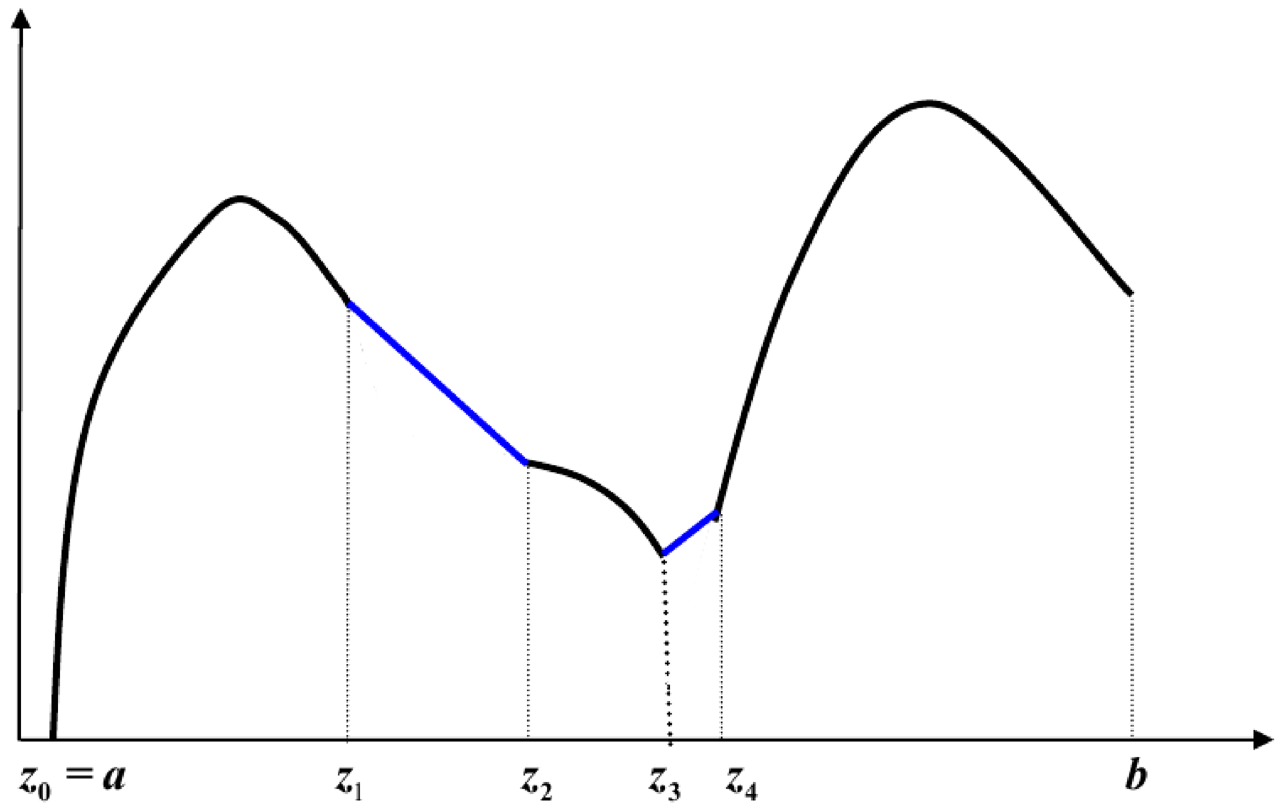

Fig. 1.2 Modification on the interval where $Lg(z) > 0$,
$Lg_{]z_1,z_2[}(z) = 0$ on $]z_1, z_2[$.
Nonstandard points: z_1, z_2, z_4, where $L_1g(z) > 0$
If $z_3 = \tilde{z}$ then for some α it is standard and for some – nonstandard.
The point z_6 where $L_1g(z_6) < 0$ is standard.

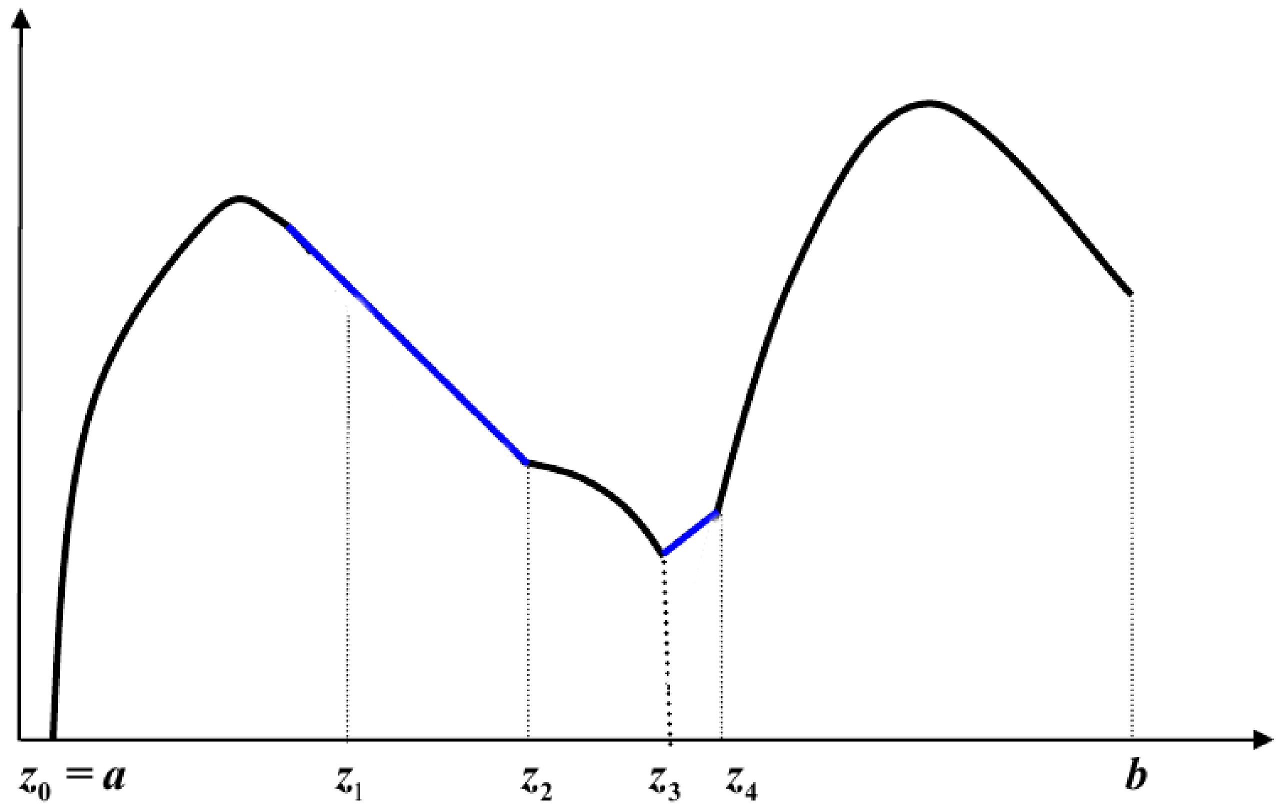

Fig. 1.3 Modification on $[a, z_2]$ at point z_1 where $L_1 g(z_1) > 0$. Smooth fitting at the left point z_7, $Lg_{]z_7,z_2[}(z) = 0$ on $]z_7, z_2[$.

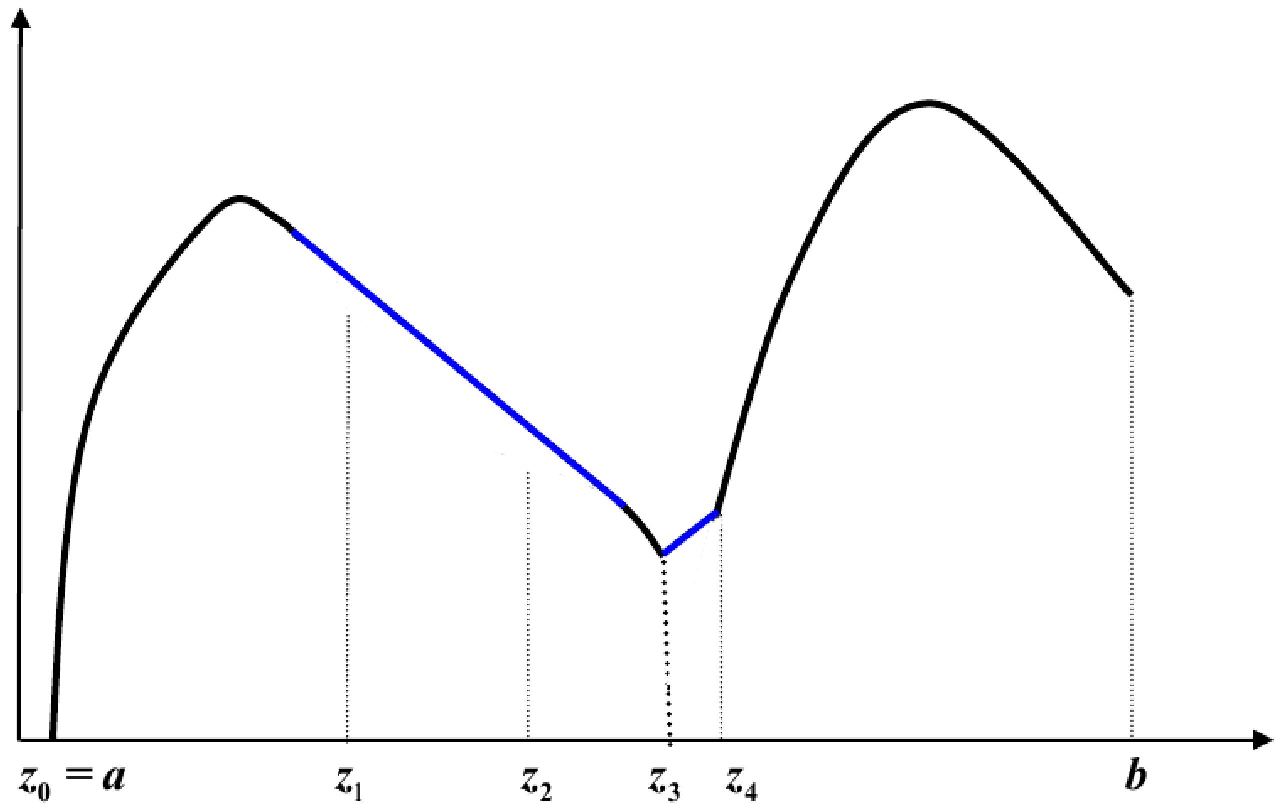

Fig. 1.4 Modification on $[a, z_3]$ at point z_2 where $L_1 g(z_2) > 0$.
Generalized smooth fitting at the left point z_6,
Smooth fitting at the left point z_8.
$Lg_{]z_6,z_8[}(z) = 0$ on $]z_6, z_8[$.

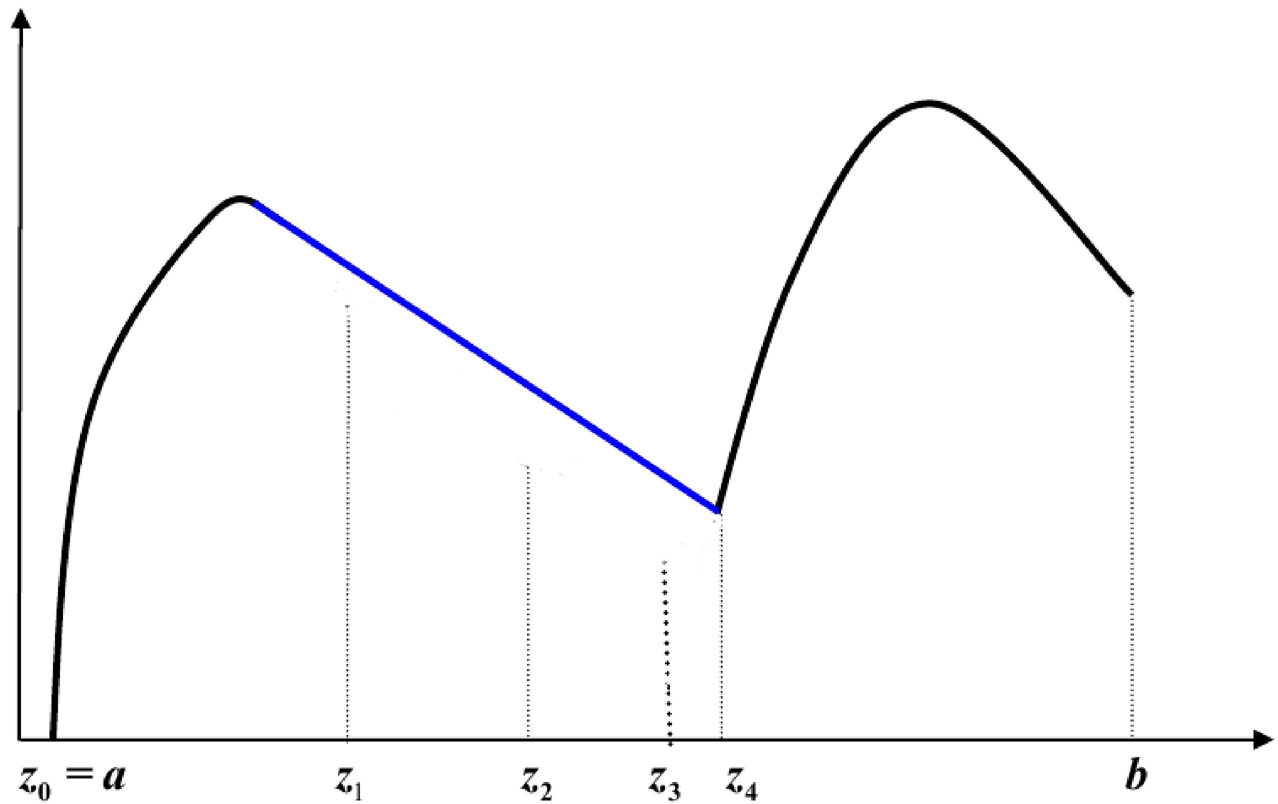

Fig. 1.5 Modification on $[a, z_4]$ at point z_3 where $L_1 g(z_3) > 0$.
(Generalized) smooth fitting at the left point z_9.
$Lg_{]z_9,z_4[}(z) = 0$ on $]z_9, z_4[$, $L_1 g_{]z_9,z_4[}(z_3) = 0$.
Different pictures for different $\alpha(z_3)$.

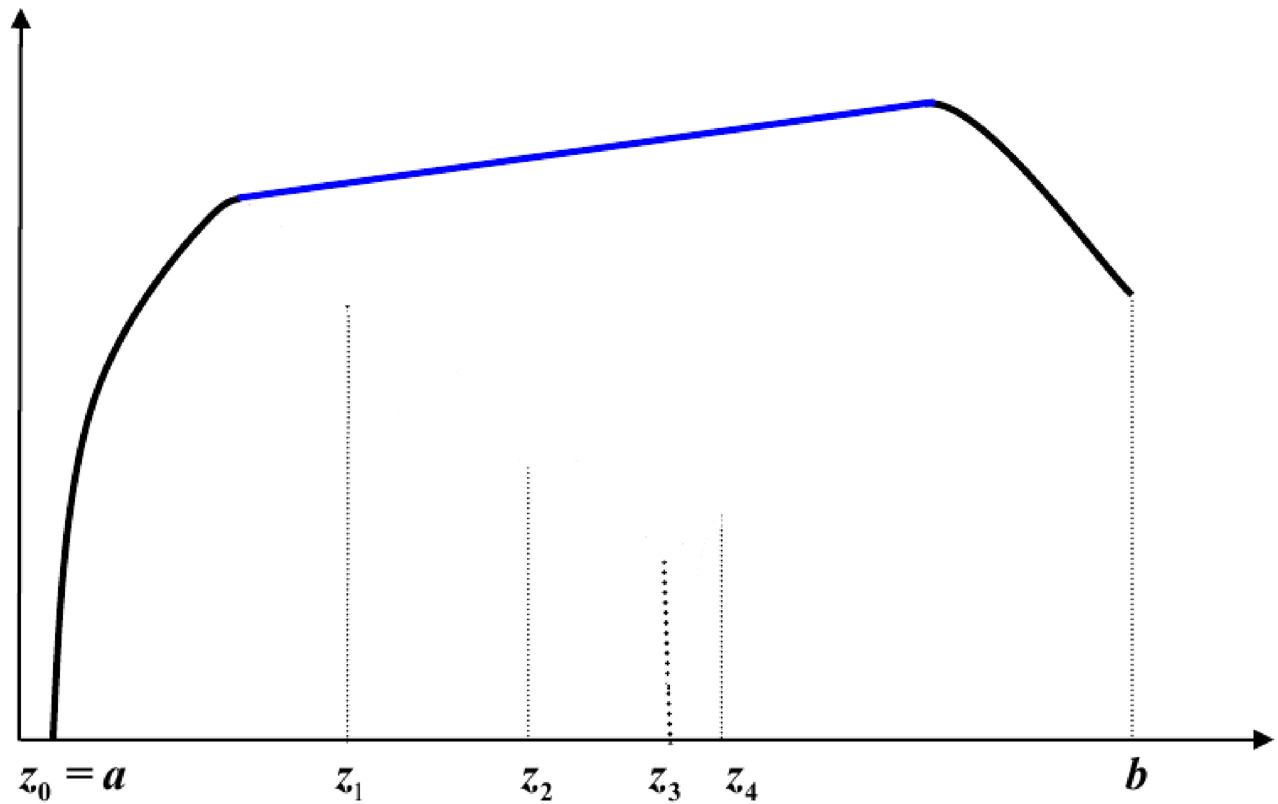

Fig. 1.6 Modification on $[a, b]$ at point z_4 where $L_1 g(z_4) > 0$.
Generalized smooth fitting at the left point z_6.
Smooth fitting at the right point z_{10}.
$Lg_{]z_6,z_{10}[}(z) = 0$ on $]z_6, z_{10}[$, $L_1 g_{]z_6,z_{10}[}(z_3) = 0$.
Different pictures for different $\alpha(z_3)$.

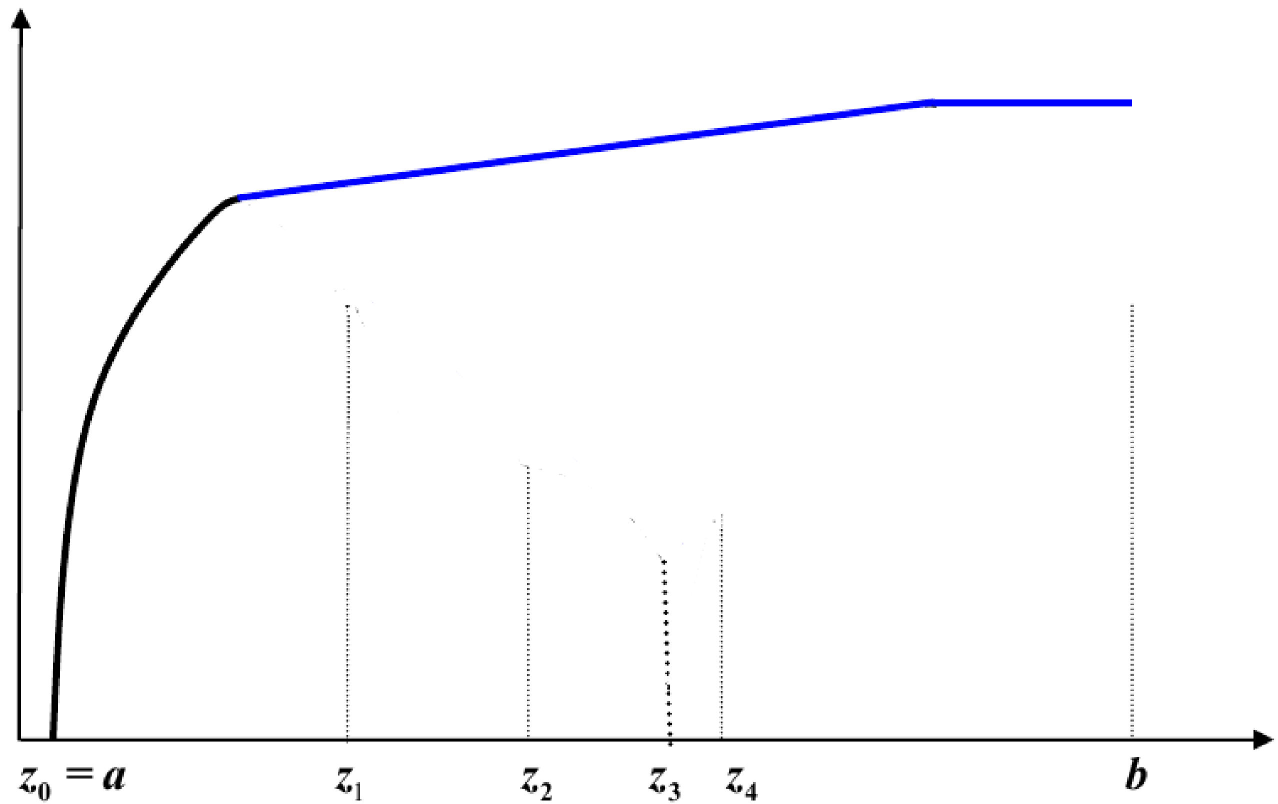

Fig. 1.7 Modification at point b in case of reflection.

3.2. One-dimensional diffusion

Homogeneous in time strong Markov process (family) $Z = (Z_t)_{t\geq 0}$ in $X \bigcup e$, e is an absorbing state and $X =]a, b[$, $-\infty \leq a < b \leq +\infty$.

$\sigma(z) \geq 0$ – diffusion coeff., $m(z)$ – drift coeff., $\rho(z) \geq 0$ – killing intensity.

I) $\forall z \in]a, b[\ \exists \varepsilon > 0 : \displaystyle\int\limits_{z-\varepsilon}^{z+\varepsilon} \frac{1 + |m(u)|}{\sigma^2(u)} du < \infty$, – i.e. diffusion is regular,

II) $\exists$ a finite (possibly empty) set $A^0 \subset]a, b[$ with a partial reflection, i. e.

$$\mathbf{P}_z[Z_t > z] \to \frac{1 + \alpha(z)}{2} \text{ as } t \to 0,$$

where $-1 < \alpha(z) < 1$, $\alpha(z) \neq 0$ for $z \in A^0$, $\alpha(z) = 0$ for $z \notin A^0$.

III) each point a and b is either natural (it can not be reached during the finite time) or reflecting or adsorbing.

Measurable function $c(z)$ — cost of observation, $c(e) = 0$.

We define also the following two operators:

$$Lf(z) := \frac{\sigma^2(z)}{2}\frac{d^2}{dz^2}f(z) + m(z)\frac{d}{dz}f(z) - \rho(z)f(z) - c(z),$$

$$L_1 f(z) := (1+\alpha(z))f'_+(z) - (1-\alpha(z))f'_-(z),$$

$\alpha(z)$ was defined in II), $f'_-(z)$ is the left, $f'_+(z)$ is the right derivative of $f(z)$.

$$Lg_{]c,d[}(z) = 0 \ \text{ for } z \in]c,d[, \ z \notin A_0, \qquad L_1 g_{]c,d[}(z) = 0 \ \text{ for } z \in]c,d[\bigcap A_0, \tag{1}$$

Algorithm for general diffusion and functions from $\mathcal{C}$ is absolutely the same as for Brownian motion.

The difference is that the operator L is different.

Instead of

$$Lf(z) := \frac{1}{2}\frac{d^2}{dz^2}f(z)$$

we have

$$Lf(z) := \frac{\sigma^2(z)}{2}\frac{d^2}{dz^2}f(z) + m(z)\frac{d}{dz}f(z) - \rho(z)f(z) - c(z).$$

The algoriphm works universally:

for the case with and without discounting depending on the state of the process,

with and without the cost of observation depending on the state of the process.

The payoff function and its derivative may have a finite number of discontinuities.

Diffusion may have finite number of points of partial reflection.

We do not need to "guess" about the structure of the stopping set,

we do not need a verification theorem,

we do not need the Bellman equation.

We simply modify sequentially the payoff function and obtain as a result the value function.

4. One-dimensional diffusion and its maximum (family)

Markov process (family) $Z_t = (Y_t, S_t)$ with initial point (y, s), where $y \le s$, Y_s is one-dimensional diffusion from the previous section, $S_t = \max[s, \sup\limits_{0 \le v \le t} Y_v]$.

$$V(y, s, \tau) = E_{y,s}\left[g(Z_\tau) - \int\limits_0^\tau c(Z_s) ds \right], \qquad V(y, s) = \sup_\tau V(y, s, \tau).$$

Peskir considered the case $g(y, s) = s$, $c(y, s) = c(y)$. He "guessed" that there exists $f(s) < s$ such that for given s it is optimal to stop for $y \le f(s)$ and to continue for $y > f(s)$. He "guessed" that on the line $y = f(s)$ the value function must be smooth. Using the necessary condition $\left.\frac{d}{ds} V(y, s)\right|_{y=s} = 0$ he obtained the differential equation for $f(s)$.

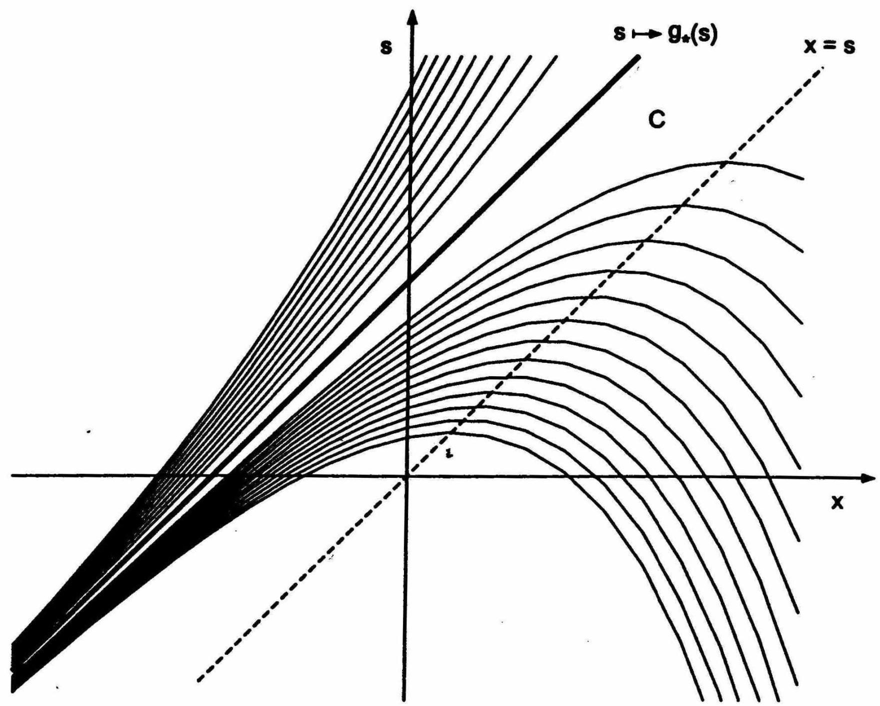

Figure IV.11: A computer drawing of solutions of the differential equation (13.2.22) in the case when $\rho \equiv 0$, $\sigma \equiv 1$ (thus $L(x) = x$) and $c \equiv 1/2$. The bold line $s \mapsto g_*(s)$ is the *maximal admissible* solution. (In this particular case $s \mapsto g_*(s)$ is a linear function.) By the maximality principle proved below, this solution is the optimal stopping boundary (the stopping time τ_* from (13.2.8) is optimal for the problem (13.1.4)).

Let $\bar{g}(y,s)$ for fixed s be the value function (v.f.) for the optimal stopping problem with an absorbtion at $y=s$ and with the payoff function $g(y,s)$.

$\bar{g}(y,s)$ is a modification of $g(y,s)$.

Let $\hat{g}(y,s,a)$ for fixed s be v.f. for optimal stopping problem with an absorbtion at $y=s$ and with the payoff function $\bar{g}(y,s)$ for $y<s$, $\hat{g}(s,s,a)=a>s$.

According to [2] $\hat{g}(y,s,a)$ is obtained from $\bar{g}(y,s)$ by generalized smooth fitting.

$V(y,s)=\hat{g}(y,s,V(s))$ where $V(s)=V(s,s)$. So, the smooth fitting along y is obtained not from intuition, but from modification. Using the necessary condition $\frac{d}{ds}\hat{g}(y,s,V(s))\Big|_{y=s}=0$ we obtain the differential equation for $V(s)$.

Let $V_{s_0}(s)$ be the solution of this equation for $s<s_0$ with initial condition $V_{s_0}(s_0)=\bar{g}(s_0,s_0)$. Then $\hat{g}(y,s,V_{s_0}(s)$ is the value function for the initial problem with absorbtion at (s_0,s_0).

$$\lim_{s_0\to b} V(y,s)=\hat{g}(y,s,V_{s_0}(s).$$

Thank you for the attention