Mean-Variance Portfolio Selection with Uncertain Drift and Volatility

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Model

The randomness is described by a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where a d-dim Brownian motion W_t is defined and $\mathcal{F}_t = \mathcal{F}_t^W$. The market consists of $n+1$ assets traded in a finite time horizon $[0, T]$. The first asset is a risk-free one with price $S_t^0 = e^{\int_0^t r_s ds}$. The other assets are risky

$dS_t^i = \mu_t^i S_t^i dt + \sum_{j=1}^d \sigma_t^{i,j} S_t^i dW_t^j.$

Notice that $\mu^i, \sigma^{i,j}$ are all $\mathcal{F}_t^W$-adapted. But the corresponding (deterministic functional) may not be known, which brings uncertainty into the model. The uncertainty is generally described as $(b_t, \sigma_t) \in A_t$ for some uniformly bounded set-valued process $A_t : (t, \omega) \mapsto \mathbb{R}^n \times \mathbb{R}^{n \times d}$. We denote the set of all these $(b_., \sigma_.)$ as Γ. We model an investment as a self-financing portfolio $\pi_.$, which is adapted and square-integrable. Its wealth process $X_.$ satisfies

$dX_t = [r_t X_t + \pi_t' b_t] dt + \pi_t' \sigma_t dW_t.$

Problem

Hence the market parameters consist of $(T, r_\cdot, W_\cdot, S_\cdot, A_\cdot)$ other than the probability space, which satisfies

Assumption

(1) $r_\cdot$ is deterministic and bounded, $T \in \mathbb{R}^+$ is deterministic.

(2) $W_\cdot$ is a standard d-dim Brownian motion.

(3) A_t is an $\mathcal{F}_t$-adapted, uniformly bounded set-valued process in $\mathbb{R}^n \times \mathbb{R}^{n \times d}$.

(4) S_t is generated by the SDE for some adapted process with $(b_t, \sigma_t) \in A_t$.

We aims at the search of admissible portfolios to optimize the bi-objective

$$\inf_{\pi} \sup_{(b,\sigma)\in\Gamma} (\mathrm{Var}(X_T), -E[X_T]).$$

Problem Setting

If there is no uncertainty on $b.$, then $A_t = \{b_t\} \times A_t^\sigma$, for some uniformly bounded, $\mathcal{F}_t^W$-adapted set-valued process $A_t^\sigma : (t, \omega) \mapsto \mathbb{R}^{n \times d}$. We focus on this case in this section. In this case,

$$\mathbb{E}[X_T] = x_0 + \int_0^T \mathbb{E}[r_t X_t + \pi_t' b] dt.$$

If r_t is deterministic, we can see that $\mathbb{E}[X_T]$ does not depend on σ, hence we can formulate the mean-variance problem as $\inf_\pi \sup_\sigma \mathbb{E}[(X_T - \lambda)^2]$. We assume $r \equiv 0$ without loss of generality.

Define $Y_t = X_t - \lambda$, then the problem turns to

$$\begin{aligned} &\inf_\pi \sup_\sigma \quad \mathbb{E}[Y_T^2] \\ &s.t. \qquad dY_t = \pi_t' b_t dt + \pi_t' \sigma_t dW_t, \;\; Y_0 = y_0 := x_0 - \lambda. \end{aligned} \tag{1}$$

It is easy to see that for $y_0 = 0$, the optimal portfolio is $\pi^* = 0$, and the optimal value is 0.

Proposition

For any π, define $\hat{\pi}_t = \pi_t \mathbf{1}_{t \leq \tau}$ with $\tau = \inf\{t \in [0, T] : Y_t = 0\}$ (with the convention $\inf \emptyset = +\infty$), then $\hat{\pi}$ is better than π, and strictly better if $\pi \not\equiv \hat{\pi}$.

This proposition shows that we only need to consider the portfolio $\pi_t = y_t u_t$ for some process $u.$, for which the objective value will be $c(\pi) y_0^2$. Hence the optimal value must be in the form $P y_0^2$ for some constant P.

Hamiltonian and BSDE

Denote

$$H(P,p,u,\sigma) := Pu'\sigma\sigma'u + 2(\sigma p + Pb)'u,$$

$$H(P,p,u) = \sup_\sigma H(P,p,u,\sigma),$$

$$F_1(P,p) = \inf_u H(P,p,u).$$

Then $H(P,p,u,\sigma) \le H(P,p,u)$ and $H(P,p,u) \ge F_1(P,p)$ for any P,p,u,σ.

Define

$$dP_t = -F_1(P_t,p_t)dt + p_t dW_t,\ P_T = 1.$$

Solution 1

Then for any control $\pi_t = u_t Y_t$, denote Y_t^{*u} as the state process for π under $\sigma^*(u_t)$, then

$$\begin{aligned}
d\mathbb{E}[P_t Y_t^2] &= \mathbb{E}\left[P_t \pi_t' \sigma_t \sigma_t' \pi_t + 2 P_t Y_t b_t' \pi_t + 2 Y_t p_t' \sigma_t' \pi_t - Y_t^2 F_1(P_t, p_t)\right] dt \\
\mathbb{E}[Y_T^2] &= P_0 Y_0^2 + \mathbb{E}\int_0^T Y_t^2 \left[H(P_t, p_t, u_t, \sigma_t) - F_1(P_t, p_t)\right] dt \\
\sup_\sigma \mathbb{E}[Y_T^2] &= P_0 Y_0^2 + \sup_\sigma \mathbb{E}\int_0^T Y_t^2 \left[H(P_t, p_t, u_t, \sigma_t) - F_1(P_t, p_t)\right] dt \\
&\geq P_0 Y_0^2 + \mathbb{E}\int_0^T (Y_t^{*u})^2 \left[H(P_t, p_t, u_t) - F_1(P_t, p_t)\right] dt \\
&\geq P_0 Y_0^2
\end{aligned}$$

Hence $\inf_u \sup_\sigma \mathbb{E}[Y_T^2] \geq P_0 Y_0^2$.

Solution 2

$$
\begin{aligned}
\sup_{\sigma} \mathbb{E}[Y_T^2] &= P_0\, Y_0^2 + \sup_{\sigma} \mathbb{E} \int_0^T Y_t^2 \left[H(P_t, p_t, u_t, \sigma_t) - F_1(P_t, p_t)\right] dt \\
&\leq P_0\, Y_0^2 + \sup_{\sigma} \mathbb{E} \int_0^T Y_t^2 \left[H(P_t, p_t, u_t) - F_1(P_t, p_t)\right] dt \\
\inf_{u} \sup_{\sigma} \mathbb{E}[Y_T^2] &\leq P_0\, Y_0^2 + \inf_{u} \sup_{\sigma} \mathbb{E} \int_0^T Y_t^2 \left[H(P_t, p_t, u_t) - F_1(P_t, p_t)\right] dt \\
&\leq P_0\, Y_0^2 + \inf_{u} \sup_{\sigma} \mathbb{E} \int_0^T Y_t^2 \left[H(P_t, p_t, u_t^*) - F_1(P_t, p_t)\right] dt \\
&= P_0\, Y_0^2.
\end{aligned}
$$

where $\pi_t^* = u_t^*\, Y_t^*$ achieves all equalities.

Main result 1

To summarize the result, we have the following theorem.

Theorem

The optimal portfolio is $\pi_t^* = u_t^*(X_t - \lambda)$*, and the optimal value of the problem is* $V = P_0(x_0 - \lambda)^2$*, where* (P, p) *is the unique solution of BSDE*

$$\begin{cases} dP_t = -F_1(P_t, p_t)dt + p_t dW_t, \\ P_T = 1 \end{cases}$$

and $u^*(t) = \operatorname{argmin} H(P_t, p_t)$*.*

Uncertain drift and volatility; Problem Setting 1

In this case, $\mathbb{E}[X_T]$ is also unknown, so we cannot formulate the mean-variance as to minimize $\mathbb{E}[X_T^2 - 2\lambda X_T]$.

In this case, we turn to study the generalization of an equivalent form of mean-variance problem. Recall that when $\mathbb{E}[X_T - x_0 e^{\int_0^T rds}] \geq 0$, the mean-variance problem is equivalent to maximize the Sharpe ratio $\frac{E[X_T - x_0 e^{\int_0^T rds}]}{\sqrt{\mathrm{Var}(X_T)}}$. In this circumstance, we take the Sharpe ratio (or its square) as our objective, and consider the problem

$$\inf_{\pi} \sup_{\sigma} \frac{\mathrm{Var}\left(X_T - x_0 e^{\int_0^T rds}\right)}{(\mathbb{E}[X_T - x_0 e^{\int_0^T rds}])^2},$$

where $X_T - x_0 e^{\int_0^T rds}$ is the excess return against the risk-free investment.

Problem Setting 2

Denote $Y_t = X_t - x_0 e^{\int_0^t r ds}$, then

$$dY_t = [rY_t + \pi_t' b]dt + \pi_t' \sigma dW_t, \quad Y_0 = 0.$$

The problem can be written as

$$\inf_{\pi} \sup_{\sigma} \frac{\text{Var}(Y_T)}{(\mathbb{E}[Y_T])^2} = \frac{\mathbb{E} Y_T^2}{(\mathbb{E}[Y_T])^2} - 1, \quad s.t. \inf_{b,\sigma} \mathbb{E} Y^2 \geq 1. \tag{2}$$

It is easy to see that the constraint is nothing but $Y \not\equiv 0$, so that the Sharpe ration is well defined.

This problem is strongly connected to the following problem:

$$\inf_{\pi} \sup_{\sigma} \mathbb{E}[Y_T^2] - \lambda(\mathbb{E} Y_T)^2, \quad s.t. \inf_{b,\sigma} \mathbb{E} Y_T^2 \geq 1. \tag{3}$$

Connection of 2 problems

Proposition

Denote the optimal value for Problem (2) as R. Denote $\Gamma = \{\pi : \inf_{b,\sigma} \mathbb{E}[Y_T^2] \geq 1\}$. Then

(i) For any $\lambda < R + 1$, $\inf_{\pi\in\Gamma} \sup_{b,\sigma} \mathbb{E}\, Y_T^2 - \lambda(\mathbb{E}\, Y_T)^2 \geq 0$.

(ii) If $\sup_{b,\sigma} \mathbb{E}\, Y_T^2 - \lambda(\mathbb{E}\, Y_T)^2 > 0$ for any $\pi \in \Gamma$, then $\lambda \leq R + 1$.

Hamiltonian and Assumption

Now we suppose $r \equiv 0$, and turn to Problem (3) with a fixed number $\lambda > 1$. Define

$$\begin{aligned} H(x, \pi, \sigma, b) &:= \pi'\sigma\sigma'\pi + 2x\pi' b, \\ H(x, \pi) &:= \sup_{b,\sigma} H(x, \pi, \sigma, b) \\ H(x) &:= \inf_{\pi} H(x, \pi). \end{aligned}$$

By changing variable $\pi \to x\pi$, we can see that $H(x) = c_t x^2$, where $c_t = \inf_\pi \sup_{b,\sigma} \pi'\sigma\sigma\pi + 2\pi' b$.

Assumption

$\int_0^T c_s ds > -\infty$.

Riccati equation and its solvability

Define the function $P_t = e^{\int_t^T c_s ds}$, which satisfies the ODE

$$dP_t = -c_t P_t dt, P_T = 1.$$

The following Riccati equation will be critical for Problem (3):

$$\begin{cases} dQ_t = \left[-2c_t Q_t - c_t e^{-\int_t^T c_s ds} Q_t^2\right] dt, \\ Q_T = -\lambda. \end{cases} \tag{4}$$

Lemma

The ODE (4) admits a finite solution in the interval $t \in [t_0, T]$ *if and only if* $\lambda < \frac{1}{1 - e^{\int_{t_0}^T c_s ds}}$.

Result for Problem (3)

Theorem

(1) If $\lambda < \frac{1}{1-e^{\int_0^T c_s ds}}$, *then*

$$\sup_{\sigma} J(\pi, \sigma, b) > 0, \quad \forall \pi \neq 0.$$

(2) If $\lambda \geq \frac{1}{1-e^{\int_0^T c_s ds}}$, *then*

$$\inf_{\pi} \sup_{b,\sigma} J(\pi, \sigma, b) = -\infty.$$

Problem (2)

From the last section, we know the optimal value for (2) is

$$R = \frac{1}{1 - e^{\int_0^T c_s ds}} - 1.$$

Proposition 4 does not provide any information on how to get the optimal π for Problem (2). To find the optimal portfolio, we turn to another approach.

Main Result 2

Theorem

Let us suppose that for each b_t, σ_t*, there exists a* θ_t *such that* $\sigma_t \theta_t = b_t$*. Then the optimal portfolio is:*

$$\pi_t^* = \{\mu E[\eta_{\theta^*}(T)^2] - Y_t^*\}(\sigma_t^*(\sigma_t^*)')^- b_t^*,$$

where

$$\theta_t^* = \mathrm{argmin}_{\sigma_t v = b_t} |v|^2,$$

and

$$\eta_{\theta^*}(t) = e^{-\int_0^t \theta_s^{*\,2}/2 ds - \int_0^t \theta_s^{*\prime} dW_s}.$$

Proof.

We have

$$\inf_{\pi} \sup_{b,\sigma} \frac{\mathbb{E}[Y_T^2]}{(\mathbb{E}[Y_T])^2} = \sup_{b,\sigma} \inf_{\pi} \frac{\mathbb{E}[Y_T^2]}{(\mathbb{E}[Y_T])^2}$$

if and only if

$$\inf_{\pi} \sup_{b_t,\sigma_t} \pi'\sigma\sigma'\pi + 2\pi' b = \sup_{b_t,\sigma_t} \inf_{\pi} \pi'\sigma\sigma'\pi + 2\pi' b.$$

The last swap holds if for each b_t, σ_t, there exists a θ_t such that $\sigma_t\theta_t = b_t$. $\square$