RAMSEY RULES AND YIELDS CURVE DYNAMICS.

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OUTLINE OF THE TALK

OUTLINE

MOTIVATIONS : DISCOUNTED LONG TERM PROJECT

- Embedded long term interest rate risk in longevity-linked securities (maturity up to $30-50$ years.) Because of the lack of liquidity for long horizon, the standard financial point of view cannot be easily extended.
- Abundant literature on the economic aspects of long-term policy-making (Ekeland, Gollier, Weitzman...), often motivated by ecological issues (Hourcade & Lecocq).
  - Main question for cost-benefit analysis (Stern review)
  - Long term Horizon
  - More specific to ecological issues : substitutability between consumption good and environment

Ramsey rule: a link between consumption and discounting

- Computation today of a long term discount factor $R_0(T)$.
- A representative agent with:
  - u utility function for representative agent, often $u(c) = c^{1-\gamma}/(1-\gamma)$
  - β pure time preference parameter
  - c aggregate consumption. Often a priori hypothesis are made on the form of the consumption function.
- Ramsey rule:

$$R_0(T) = \beta - \frac{1}{T} \ln \mathbb{E}\left[\frac{u'(c_T)}{u'(c_0)}\right].$$

- Very popular particular case (Ramsey, 1928):

$$R_0(T) = \beta + \gamma g,$$

  β pure time preference parameter, γ risk aversion, g consumption growth rate.
- Example : Stern review on climate change (2006), with $\gamma = 1$, $g = 1.3\%$, $\beta = 0.1\% \to R_0(T) = 1.4\%$.
- Or $\gamma = 1, 5$, $g = 2\%$, $\beta = 0.1\% \to R_0(T) = 3.6\%$
- Controversy between economists concerning parameters values.
  $R_0(T) = 1.4\%$: \$ 1 million in 100 years $\to$ \$ 250,000 today.
  $R_0(T) = 3.5\%$: \$ 1 million in 100 years $\to$ \$ 32,000 today.
- small β= intergenerational equity

Cox-Ingersoll-Ross (1985)

Equilibrium approach that determines the interest rate **endogenously**.

- single consumption good
- the production process follows a diffusion whose coefficients depend on an exogeneous stochastic factor Y influencing the economy.
- investors are indifferent between an investment in the production opportunity and the risk-free instrument.
- all investors are identical, and share the same stochastic preference structure $U(t, c_t, Y_t)$
- classic CRRA utility function + CIR diffusion for Y
  $\Rightarrow$ CIR dynamic for the risk free rate

Aggregation of the heterogeneity

Jouini et al. (2008) : heterogeneous beliefs and anticipations, heterogeous time preference rates.

$$\begin{aligned} \text{Belief dispersions} &\Rightarrow \text{additional risk or uncertainty} \\ &\Rightarrow \text{more saving} \\ &\Rightarrow \text{lower discount rate} \end{aligned}$$

Aggregation of individual beliefs and time-preferences

- the more heteregenous are the agents, the lower is the discount rate.
- the relevant asymptotic behavior in the long term is the one with the lowest discount rate.
- the asymptotic equilibrium discount rate is given by the lowest individual discount rate.

AIM : Extend the economic framework

- by taking into account the existence of a financial market
- by a dynamic and stochastic point of view

The method

- Introduce the financial market model (incomplete market)
- Maximize the representative agent's utility on the aggregate consumption.
- → Link with the yield curve, extension of the Ramsey rule.
- → Link with Growth Optimal Portfolio. (Platen & Heath, 2006)

THE GROWTH OPTIMAL PORTFOLIO

The GOP is the portfolio maximizing the expected log utility from terminal wealth over all positive portfolio.

- Approximation of the GOP by a diversified world stock index.
- The GOP is a particularly robust portfolio on the long term
  The GOP outperforms the long term growth rate

$$\limsup_{T \to +\infty} \frac{1}{T} \ln \left(\frac{X_T}{X_0} \right)$$

- The benchmark approach (Platen) uses the GOP as a benchmark/numeraire and historical probability measure as the pricing measure $\rightarrow$ it is possible to use the GOP for pricing zeros-coupons.

$\Rightarrow$ The GOP seems to be a useful tool for the study of long term interest rate.

OUTLINE

THE FINANCIAL MARKET

- Filtered probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.
- N-dimensional Brownian motion

Market Parameters : Incomplete market

- M risky assets, $M \leq N$.
- $(r_t)_{t\geq 0}$, $(\theta_t)_{t\geq 0}$, $(\sigma_t)_{t\geq 0}$ adapted processes.
- $r_t \geq 0$ spot rate.
- θ_t market price of risk process.
- σ_t volatility process $M \times N$. $\sigma_t \sigma_t^{\mathbf{T}}$ invertible.

THE REPRESENTATIVE AGENT

- Representative agent, strategy (π, c).
  - $c(.)$: **consumption rate**.
  - $\pi(.)$: fractions of the wealth invested in the risky asset. We set

$$\kappa_t := \sigma_t^{\mathbf{T}} \pi_t.$$

- Constraints on the portfolio $\Rightarrow$ Incompleteness of the market. $\kappa_t \in \mathcal{K}_t$ where $\mathcal{K}_t$ adapted subvector spaces in $\mathbb{R}^N$. Typically $\mathcal{K}_t = \sigma_t(\mathbb{R}^M)$, $M \leq N$.
- Self financing positive wealth process $X^{x,c,\kappa}(.)$ starting from $X_0^{x,c,\kappa} = x > 0$:

$$dX_t^{x,c,\kappa} = -c_t dt + X_t^{x,c,\kappa}(r_t dt + \kappa_t(dW_t + \theta_t dt)),$$

Growth optimal portfolio or Market Numeraire

State price density: A process Y is said to be a state price density (or adjoint process) if for any $\kappa \in \mathcal{K}$, $YX^{x,0,\kappa}$ is a local martingale $\Rightarrow$ there exists $\nu \in \mathcal{K}^{\perp}$:

$$\frac{dY_t^{\nu}}{Y_t^{\nu}} = -r_t dt - (\nu_t + \theta_t).dW_t, \ \nu_t \in \mathcal{K}_t^{\perp}$$

- The Growth Optimal Portfolio intrinsic to the market

$$G_t^* = \exp\left(\int_0^t r_s + \frac{1}{2}||\theta_s||^2 ds + \int_0^t \langle \theta_s, dW_s \rangle\right).$$

- The **Growth Optimal Portfolio** is the portfolio maximizing the expected log utility from terminal wealth over all strictly positive portfolio (also known **as market numeraire**)
- The GOP is the inverse of the "minimal" state price density

Utility maximisation of the representative agent

- **Representative agent** start with an initial wealth $x > 0$.
- Preference structure $(U(t,.), V(.))$, increasing and concave
- **Primal problem** : the representative agent maximises his expected utility from terminal wealth and consumption over all (κ, c) admissible strategies [Karatzas & Shreve,1998].

$$\max_{(\kappa,c)\in\mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\Big[\int_0^{T_H} \underbrace{U(t,c_t)}_{example: U(t,c)=e^{-\beta t}u(c)} dt + V(X^{x,c,\kappa}_{T_H})\Big].$$

THE DUAL PROBLEM

- Y^{ν} = **state price density process**

$$\frac{dY_t^{\nu}(y)}{Y_t^{\nu}(y)} = -r_t dt - (\nu_t + \theta_t) dW_t, Y_0^{\nu}(y) = y, \nu_t \in \mathcal{K}_t^{\perp}.$$

- $Y_t^{\nu} X_t^{x,c,\kappa} + \int_0^t Y_s^{\nu} c_s ds$ local martingale.
- **Dual problem** :

$$\inf_{\nu \in \mathcal{K}^{\perp}} \mathbb{E}^{\mathbb{P}}[\int_0^{T_H} \tilde{U}(t, Y_t^{\nu}(y))dt + \tilde{V}(Y_T^{\nu}(y))].$$

where $\tilde{U}$ and $\tilde{V}$ Fenchel transform of V and V, i.e.

$$\tilde{U}(t,y) = \inf_{c>0}\{U(t,c) - cy\}.$$

SOLUTION OF THE UTILITY MAXIMISATION PROBLEM

All optimal processes are depending on the horizon T_H of the problem, through the solution of the dual problem $\nu^{*,H}$.

- Solution of the primal problem : $(X^{*,H}(x), c^{*,H}(c_0))$
- Solution of the dual problem : $\nu^{*,H} \rightarrow Y^{\nu^{*,H}}(.) =: Y^{*,H}$
- **Optimal consumption path**:

$$Y_t^{*,H}(y) = U_c(t, c_t^{*,H}(c_0)) \text{ and } Y_{T_H}^{\nu^*}(y) = V_x(X_{T_H}^{*,H}(x)).$$

- Budget constraint :

$$y = U_c(0, c_0) = V_x(x).$$

EXAMPLE

Illustration of the time-unconsistency on the example of CRRA utility function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with $0 < \gamma$.

- Incomplete market with 2 sources of uncertainty, generated by a 2 dimensional-Brownian motion (W^1, W^2).
  - One riskless asset with dynamics $dS_t^0 = r_t S_t^0 dt$ where the short rate dynamics follows an Ornstein-Uhlenbeck process:

$$dr_t = a(b - r_t)dt - \alpha dW_t^2.$$

  - One risky asset

$$\frac{dS_t}{S_t} = r_t dt + \sigma(dW_t^1 + \eta_t dt)$$

    $\sigma > 0$ constant, (η_t) deterministic.
- The state price density processes satisfy

$$\frac{dY_t^\nu}{Y_t^\nu} = -r_t dt + (\nu_t dW_t^2 + \eta_t dW_t^1)$$

  with $\nu \in \mathcal{R}^\perp$ the filtration generated by W^2
- The optimal state price is determined by

$$\nu_s^{*,T_H} = (\gamma - 1)\frac{\sigma}{a}(1 - e^{a(T_H - s)}).$$

- Link between the state price density process and the **marginal utility from consumption**.

$$\frac{U_c(t, c_t^{*,H}(c_0))}{U_c(0, c_0)} = \exp(-\int_0^t r_s ds)\mathcal{E}(-\int_0^t \langle \theta_s + \nu_s^{*,H}(y), dW_s \rangle).$$

- We take the expectation under the historical probability :

$$\mathbb{E}^{\mathbb{P}}\left[\frac{U_c(t, c_t^{*,H}(c_0))}{U_c(0, c_0)}\right] = \mathbb{E}^{\mathbb{P}}\left[\exp(-\int_0^t r_s ds)\mathcal{E}(-\int_0^t \langle \theta_s + \nu_s^{*,H}(y), dW_s \rangle)\right]$$

- or more dynamically, thanks to the flow property, for $t \leq T \leq T_H$:

$$\mathbb{E}^{\mathbb{P}}\left[\frac{U_c(T, c_T^{*,H}(c_t^{*,H}(c_0)))}{U_c(t, c_t^{*,H}(c_0))}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[Y_{t,T}^{*,H}(y)|\mathcal{F}_t\right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[\exp(-\int_t^T r_s ds)\mathcal{E}(-\int_t^T \langle \theta_s + \nu_s^{*,H}(y), dW_s \rangle)|\mathcal{F}_t\right].$$

with $Y_{t,T}^{*,H}(y) := \frac{Y_T^{*,H}(y)}{Y_t^{*,H}(y)}$

Financial interpretation of the equilibrium yield curve I

For $t \leq T \leq T_H$

- $(B^m(t,T),\ t \leq T)$, (m for market) : price at time t of a zero coupon bond paying one unit of cash at maturity T.
- In finance, the yield curve is linked with the price of zero coupon bonds:

$$B^m(t,T) = \exp(-R^m(t,T)(T-t)).$$

AIM: Give a financial interpretation of the equilibrium yield curve given by the Ramsey rule.

Ramsey rule in complete market I

- In a complete market:
  $\nu^* = 0$, the optimal processes do not depend on T_H.

$$\mathbb{E}^{\mathbb{P}}\left[\frac{U_c(T, c_T^*(c_0))}{U_c(0, c_0)}\right] = \mathbb{E}^{\mathbb{P}}\Big[\exp(-\int_0^T r_s ds)\,\overbrace{\mathcal{E}(-\int_0^T \langle \theta_s, dW_s\rangle)}^{\frac{d\mathbb{Q}}{d\mathbb{P}}}\Big].$$

- $B^m(0, T)$ defined by

$$B^m(0, T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s ds\right)\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{1}{y} Y_T^*(y)\right]$$

where $\mathbb{Q}$ is the risk neutral probability measure (unique).

$$B^m(0, T) = \mathbb{E}^{\mathbb{P}}\left[\frac{U_c(T, c_T^*(c_0))}{U_c(0, c_0)}\right].$$

RAMSEY RULE IN COMPLETE MARKET II

- Financial yield curve:

$$R^m(0,T) = -\frac{1}{T}\ln \mathbb{E}^{\mathbb{P}}\left[\frac{U_c(T, c_T^*(c_0))}{U_c(0, c_0^*)}\right],$$

$$R^m(t,T) = -\frac{1}{T-t}\ln \mathbb{E}^{\mathbb{P}}\left[\frac{U_c(T, c_T^*(c_0))}{U_c(t, c_t^*(c_0))}|\mathcal{F}_t\right].$$

- This formula still holds in incomplete market in the case of replicable Zero coupons with an admisible self financing portfolio.
- Example : with time separable utility $U(t,c) = e^{-\beta t}u(c)$.

$$R^m(t,T) = \beta - \frac{1}{T-t}\ln \mathbb{E}^{\mathbb{P}}\left[\frac{u'(c_T^*(c_0))}{u'(c_t^*(c_0))}|\mathcal{F}_t\right].$$

Ramsey rule in complete market III

Remark: Intrinsic formulation

- Price of zero-coupons inferred from Growth Optimal Portfolio [Platen]

$$\frac{B^G(t,T)}{G_t^*} = \mathbb{E}^{\mathbb{P}}\left[\frac{1}{G_T^*}|\mathcal{F}_t\right].$$

- Ramsey rule

$$R^m(t,T) = -\frac{1}{T-t}\log \mathbb{E}^{\mathbb{P}}\left[\frac{U_c(T, c_T^*(c_0))}{U_c(t, c_t^*(c_0))}|\mathcal{F}_t\right] = \underbrace{-\frac{1}{T-t}\ln \mathbb{E}^{\mathbb{P}}\left[\frac{G_t^*}{G_T^*}|\mathcal{F}_t\right]}_{\text{Approximated with market data}}$$

INCOMPLETE MARKET

In a incomplete market :

$$\mathbb{E}^{\mathbb{P}}\left[\frac{U_c(T, c_T^{*,H}(c_0))}{U_c(t, c_t^{*,H}(c_0))}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}^{\nu^{*,H}(y)}}[\exp(-\int_t^T r_s ds)|\mathcal{F}_t].$$

$$\left.\frac{d\mathbb{Q}^{\nu^{*,H}(y)}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}(-\int_0^t \langle \theta_s + \nu_s^{*,H}(y), dW_s \rangle).$$

The "pricing" probability $\mathbb{Q}^{\nu^{*,H}(y)}$ is not universal and might depend on

- the maturity T_H
- the utility function
- the wealth y in the economy

PRICING RULE IN INCOMPLETE MARKET

Indifference pricing :

- Utility indifference price of a positive claim ζ_{T_H} = the cash amount p for which the investor is indifferent between investing optimally a certain quantity q in the claim and investing optimally in the market without the claim but endowed with an extra amount p of money:
- $(p_t)_{t\in[0,T]}$ determined by the non linear relationship $\mathcal{U}^\zeta(t, X_t - p_t, q) = \mathcal{U}(t, X_t^*), \quad$ for all $t \in [0, T_H]$. with

$$\begin{aligned}
\mathcal{U}^\zeta(t, X_t, q) \quad &:= \sup_{(\kappa,c)\in\mathcal{A}(t,X_t)} \mathbb{E}[V(X^\kappa_{T_H} + q\zeta_{T_H}) + \int_t^{T_H} U(s, c_s)ds|\mathcal{F}_t], \\
\mathcal{U}(t, X_t) \quad &:= \sup_{(\kappa,c)\in\mathcal{A}(t,X_t)} \mathbb{E}[V(X^\kappa_{T_H}) + \int_t^{T_H} U(s, c_s)ds|\mathcal{F}_t], \quad t \le T_H
\end{aligned}$$

- If $q > 0$ (resp. $q < 0$) $p =: p^b$ is a buying (resp. $p =: p^s$ is a selling) indifference price
- non-linear pricing rule and provides a price range $[p_t^b(q), p_t^s(q)]$.

Utility Davis price: a fair price for small transactions

When the agents are aware of their sensitivity to the unhedgeable risk, they can try to transact for only a little amount in the risky contract.

- Davis price or marginal utility price, which corresponds to the zero marginal rate of substitution $\widehat{p}_t$:

$$\partial_q \mathbb{E}[V(X^{*,H}_{T_H}) + q\zeta_{T_H}) + \int_t^{T_H} U(s, c^{*,H}_s) ds | \mathcal{F}_t]_{|_{q=0}} =$$

$$\partial_q \mathbb{E}[V(X^{*,H}_t - qp_t) + \int_t^{T_H} U(s, c^{*,H}_s) ds | \mathcal{F}_t]_{|_{q=0}}.$$

-

$$\widehat{p}_t = \frac{\mathbb{E}[V_x(X^{*,H}_{T_H})\zeta_{T_H} | \mathcal{F}_t]}{\mathbb{E}[V_x(X^{*,H}_{T_H}) | \mathcal{F}_t]} = \mathbb{E}(\zeta_{T_H} Y^{*,H}_{t,T_H}(y) | \mathcal{F}_t)$$

- If the maturity of the claim is $T < T_H$, $\zeta_{T_H} = \zeta_T e^{\int_T^{T_H} r_s ds}$ and

$$\widehat{p}_t = \mathbb{E}(\zeta_T Y^{*,H}_{t,T}(y) | \mathcal{F}_t).$$

RAMSEY RULE IN INCOMPLETE MARKET

- Davis price is a linear pricing rule.
- Using Davis prices means that there exists a consensus at this price for a small amount, but investors are not sure to have liquidity at this price.
- The price $B^{m,u}(t,T)$ at time t of a Zero-Coupon bond with maturity T, using Davis rule, is defined by $B^{m,u}(t,T) = \mathbb{E}^{\mathbb{P}}\left[Y^{*,H}_{t,T}(y)|\mathcal{F}_t\right]$ and satisfies

$$B^{m,u}(t,T) = \mathbb{E}^{\mathbb{P}}\left[\frac{U_c(T, c^{*,H}_T(c_0))}{U_c(t, c^{*,H}_t(c_0))}|\mathcal{F}_t\right].$$

- Ramsey rule may = yield curve “marginal indifference pricing”

$$R^{m,u}(t,T) = -\frac{1}{T-t}\ln \mathbb{E}\left[\frac{U_c(T, c^{*,H}_T(c_0))}{U_c(t, c^{*,H}_t(c_0))}|\mathcal{F}_t\right], \quad 0 \le t < T \le T_H.$$

- Acceptable for small trade
- for large trade use a second order correction term depending on the size of the trade

Dynamic of zero-coupons price and link with the GOP

- Price of zero-coupons inferred from GOP [Platen] (the GOP is used as a numeraire) :

$$\frac{B^G(t,T)}{G_t^*} = \mathbb{E}^{\mathbb{P}}\left[\frac{1}{G_T^*}|\mathcal{F}_t\right].$$

- Link with the price of zeros-coupons :

$$B^{m,u}(t,T) = B^G(t,T)\mathbb{E}^{\mathbb{Q}_T^G}\left[\mathcal{E}\left(-\int_t^T \langle \nu_s^{*,H}(y), dW_s\rangle\right)|\mathcal{F}_t\right],$$

where $\mathbb{Q}_T^G$ is similar as a forward neutral probability measure :

$$\frac{d\mathbb{Q}_T^G}{d\mathbb{P}}|_{\mathcal{F}_T} = \frac{1/G_T^*}{\mathbb{E}^{\mathbb{P}}[1/G_T^*]}.$$

OUTLINE

Dynamic utility functions from consumption and terminal wealth I

- In the presence of generalized long term uncertainty, the decision scheme must evolve: the economists agree on the necessity of a sequential decision scheme that allows to revise the first decisions according to the evolution of the knowledge and to direct experiences.
- AIM : Extend the previous results to the case of dynamic utility functions to take into account that the preferences of the agent may changes with time.
- AIM : To get rid of the dependency on the maturity T_H.
- References: Musiela & Zariphopoulou, El Karoui & Mrad (dynamic utility functions from terminal wealth), Berrier & Rogers & Tehranchi (dynamic utility functions from consumption and terminal wealth).

Dynamic utility

Definition of Dynamic Utility A progressive utility is a positive family $\mathbf{U} = \{U(t;x) : t \geq 0; x > 0\}$

- Progressivity : for any $x > 0$, $t \to U(t;x)$ is a progressive random field
- Concavity : for $t \geq 0$, $x > 0 \to U(t;x)$ is an increasing concave function.
- Inada condition : $U(:;x)$ is a C^2-function with marginal utility $U_x(:;:)$, decreasing from $+\infty$ to 0.
- Initial condition : $U(0,x) =: u$ a deterministic positive C^2-utility function with Inada condition

Consistent Dynamic utility

Let $\mathcal{A}$ be a convex family of non negative portfolios, called Test porfolios.
A $\mathcal{A}$-consistent dynamic utility system of investment and consumption is a pair of progressive dynamic utilities **U** and **V** on $\Omega \times [0, +\infty) \times \mathbb{R}^+$ with the following additional properties:

- Consistency with the test-class: For any admissible wealth process $X^{\kappa,c} \in \mathcal{A}$,

$$\mathbb{E}(V(t, X_t^{\kappa,c}) + \int_s^t U(s, c_s)ds|\mathcal{F}_s) \leq V(s, X_s^{\kappa,c}), \ \forall s \leq t \ \ a.s.$$

  That is, the process $(V(t, X_t^{\kappa,c}) + \int_s^t U(s, c_s)ds)$ is a supermartingale.
- Existence of optimal strategy: For any initial wealth $x > 0$, there exists an optimal strategy (κ^*, c^*) such that the associated non negative wealth process $X^* = X^{\kappa^*,c^*} \in \mathcal{A}$ issued from x satisfies $(V(t, X_t^*) + \int_0^t U(s, c_s^*)ds)$ is a local martingale.

Dual dynamic utility

Proposition : Pair of **Dual dynamic utility functions**.

- The conjugate utility $\tilde{U}(t,y)$ and $\tilde{V}(t,y)$ are decreasing convex stochastic flow.
- For all state price density processes $Y^{\nu}(y)$, the following process is a submartingale:

$$\tilde{V}(t, Y_t^{\nu}(y)) + \int_0^t \tilde{U}(s, Y_s^{\nu}(y))ds. \tag{1}$$

- Existence of an optimum ν^*, such that

$$\tilde{V}(t, Y_t^{\nu^*}(y)) + \int_0^t \tilde{U}(s, Y_s^{\nu^*}(y))ds$$

is a martingale.

Dynamic utility and Ramsey rule

- **Optimal consumption and wealth paths**:

$$Y_t^*(y) := Y_t^{\nu^*}(y) = U_c(t, c_t^*(c_0)) = V_x(t, X_t^*(x)).$$

with

$$\begin{aligned} y &= U_c(0, c_0) = u_c(c_0) \\ &= V_x(0, x) = v_x(x) \end{aligned}$$

- The yield curve (marginal indifference pricing) in the case of dynamic utility does not depend on the horizon T_H, even in the case of incomplete market :

$$R^{m,u}(0,T) = -\frac{1}{T} \ln \mathbb{E}^{\mathbb{P}}[\frac{U_c(T, c_T^*(c_0))}{U_c(0, c_0)}] = -\frac{1}{T} \ln \mathbb{E}^{\mathbb{P}}[\frac{Y_T^*(y)}{y}].$$

$$R^{m,u}(t,T) = -\frac{1}{T-t} \ln \mathbb{E}\left[\frac{U_c(T, c_T^*(c_0))}{U_c(t, c_t^*(c_0))} |\mathcal{F}_t\right] = -\frac{1}{T-t} \ln \mathbb{E}^{\mathbb{P}}\left[Y_{t,T}^*(y)|\mathcal{F}_t\right].$$

- Note the key role of the process $(Y_t^*(y))_{t\geq 0}$ to compute the yield curve.

CONCLUSION

- Numerics (approximation of the Growth Optimal Portfolio)
- Different beliefs of the agents
- Calibration of dynamics utilities

BIBLIOGRAPHY

- [1] Ivar Ekeland. Discounting the future: the case of climate change. Lecture Notes 2009.
- [2] Nicole El Karoui, Mohamed M'Rad : An Exact Connection between two Solvable SDEs and a Non Linear Utility Stochastic PDEs, (2012) to appear.
- [3] Christian Gollier. The consumption-based determinants of the term structure of discount rates. Mathematics and Financial Economics, 1(2):81-101, July 2007.
- [4]Elyès Jouini, Jean-Michel Marin, and Clotilde Napp. Discounting and divergence of opinion. Journal of Economic Theory, 145:830-859, 2010.
- [6] Franck Lecocq and Jean-Charles Hourcade. Le taux d'actualisation contre le principe de précaution? Leçons partir du cas des politiques climatiques.
- [7] E. Platen and D. Heath. A Benchmark Approach to Quantitative Finance. Springer Finance, Springer-Verlag, Berlin, 2006.