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Arbitrages in a progressive enlargement of filtrations before and after the default

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Je ne cherche pas à connaître les réponses, je cherche à comprendre les questions.

Confucius (Entretiens)

We consider a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ and a random time τ (i.e., a non negative finite $\mathcal{A}$-measurable random variable).

We assume that the financial market where a risky asset with price S (an $\mathbb{F}$-adapted positive process) and a riskless asset (assumed to be constant, for simplicity) are traded is arbitrage free. More precisely, we assume w.l.g. that S is a $(\mathbb{P}, \mathbb{F})$ (local) martingale.

We denote by $\mathbb{G}$ the progressively enlarged filtration of $\mathbb{F}$, i.e.,

$$\mathcal{G}_t = \cap_{s>t}\mathcal{F}_s \vee \sigma(\tau \wedge s)$$

Our goal is to detect if the knowledge of τ allows for some arbitrage, i.e. if, using $\mathbb{G}$-adapted strategies, one can make profit. We start by an elementary remark: assume that there are no arbitrages in $\mathbb{G}$. Then, roughly speaking, S would be a $(\mathbb{Q}, \mathbb{G})$ martingale for some e.m.m. $\mathbb{Q}$, hence would be also a $(\mathbb{Q}, \mathbb{F})$ martingale. In case of a complete market, this implies that any $\mathbb{F}$ martingale would be a $\mathbb{G}$ martingale. This last property is known as **immersion property**.

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Illustrative examplse

Let $dS_t = S_t \sigma dW_t$ where W is a Brownian motion.

This martingale S goes to 0 when t goes to infinity, hence the random time $\tau = \sup\{t \,:\, S_t = \sup_s S_s\}$ is well defined, and obviously leads to arbitrages:

• at time 0, buy one share of S (at price S_0), borrow S_0, then, at time τ, reimburse the loan S_0 and sell the share at price S_τ. The gain is $S_\tau - S_0 > 0$ with an initial wealth null.

• At time τ, shortsell S for a delivery at time $\tau + \epsilon$. This strategy is admissible, S being bounded above by S_τ.

One can find, in Dellacherie, Maisonneuve, Meyer (1992), *Probabilités et Potentiel, chapitres XVII-XXIV: Processus de Markov (fin), Compléments de calcul stochastique*, page 137 ***Par exemple,*** S_t ***peut représenter le cours d'une certaine action à l'instant*** t***, et*** $\tau = \sup\{t, S_t = \sup_s S_s\}$ ***est le moment idéal pour vendre son paquet d'actions. Tous les spéculateurs cherchent à connaître*** τ ***sans jamais y parvenir, d'où son nom de variable aléatoire honnête.*** For instance, S_t may represent the price of some stock at time t and τ is the optimal time to liquidate a position in that stock. Every speculator strives to know when τ will occur, without ever achieving this goal. Hence, the name of honest random variable.

We shall define general honest times later.

The problem is fully different when there are jumps

Let N be a Poisson process with intensity λ and M be its compensated martingale. Define the price process S as $dS_t = S_{t-}\varphi dM_t$ with φ a constant satisfying $\varphi > -1$, so that

$$S_t = S_0 \exp(-\lambda\varphi t + \ln(1+\varphi)N_t)\,.$$

The random time $\tau = \sup\{t \,:\, S_t = \sup_s S_s\}$ is well defined.

• If $\varphi > 0$, $S_\tau \geq S_0$ and an arbitrage opportunity is realized at time τ, with a long position in the stock. There are arbitrages after τ (shortselling)

• If $\varphi < 0$, due to continuity on right of the process, one has $S_{\tau-} = \sup S_s$ and $S_\tau < S_{\tau-}$. We shall see that there are NA before τ and there are arbitrages after τ (the price being bounded above by $S_{\tau-}$).

Admissible Portfolio and Arbitrages Opportunities

Let $(\mathbb{K})$ be one of the filtrations $\{\mathbb{F}, \mathbb{G}\}$.

We denote $(\theta \cdot S)_t = \int_0^t \theta_s dS_s$.

For $a \in \mathbb{R}_+$, an element $\theta \in L^{\mathbb{K}}(S)$ is said to be an ***a-admissible $\mathbb{K}$-strategy*** if $(\theta \cdot S)_\infty := \lim_{t\to\infty}(\theta \cdot S)_t$ exists and $V_t(0,\theta) := (\theta \cdot S)_t \geq -a$ $\mathbb{P}$-a.s. for all $t \geq 0$. We denote by $\mathcal{A}_a^{\mathbb{K}}$ the set of all a-admissible $\mathbb{K}$-strategies. We say that an element $\theta \in L^{\mathbb{K}}(S)$ is an *admissible* $\mathbb{K}$-*strategy* if $\theta \in \mathcal{A}^{\mathbb{K}} := \bigcup_{a\in\mathbb{R}_+} \mathcal{A}_a^{\mathbb{K}}$.

Various kinds of arbitrages

An element $\theta \in \mathcal{A}^{\mathbb{K}}$ yields an ***Arbitrage Opportunity*** if $V\,(0,\theta)_\infty \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}\big(V\,(0,\theta)_\infty > 0\big) > 0$. In order to avoid confusions, we shall call these arbitrages **strong arbitrages**.

If there exists no such $\theta \in \mathcal{A}^{\mathbb{K}}$ we say that the financial market $(\Omega, \mathbb{K}, \mathbb{P}; S)$ satisfies the ***No Arbitrage (NA)*** condition.

NFLVR holds in the financial market $(\Omega, \mathbb{K}, \mathbb{P}; S)$ if and only if there exists an Equivalent Martingale Measure in $\mathbb{K}$, i.e. $\mathbb{Q} \sim \mathbb{P}$ so that the process S is a $(\mathbb{Q}, \mathbb{K})$-local martingale.

A non-negative $\mathcal{K}_\infty$-measurable random variable ξ with $\mathbb{P}(\xi > 0) > 0$ yields an **Arbitrage of the First Kind** if for all $x > 0$ there exists an element $\theta^x \in \mathcal{A}_x^{\mathbb{K}}$ such that $V(x, \theta^x)_\infty := x + (\theta^x \cdot S)_\infty \geq \xi$ $\mathbb{P}$-a.s. If there exists no such random variable we say that the financial market $(\Omega, \mathbb{K}, \mathbb{P}; S)$ satisfies the *No Arbitrage of the First Kind (NA1)* condition.

We say that S satisfies **No Unbounded Profit with Bounded Risk (NUPBR)** if

$$K(S) := \{(H \cdot S)_\infty : H \in L(S) \text{ and } H \cdot S \geq -1\}$$

is bounded in $L^0(P)$.

One can prove that NAI is equivalent to NUPBR.

Theorem (Takaoka) An $\mathbb{F}$-semimartingale S **satisfies NUPBR if and only if**

$$\mathcal{L}_\sigma(S) \neq \emptyset$$

where $\mathcal{L}_\sigma(S)$ is the set of σ-densities given by

$$\mathcal{L}_\sigma(S) := \{L \in \mathcal{M}_{loc}(\mathbb{F}) : L > 0 \text{ and } LS \text{ is a sigma-martingale}\}$$

An $\mathbb{R}$-valued semimartingale X is called a sigma-martingale if there exists an $\mathbb{R}$ -valued martingale M and an M-integrable predictable $\mathbb{R}^+$-valued process φ such that $X = \varphi \cdot M$.

A strictly positive $\mathbb{K}$- **local** martingale $L = (L_t)_{t \geq 0}$ with $L_0 = 1$ and $L_\infty > 0$ $\mathbb{P}$-a.s. is said to be a *local martingale deflator in* $\mathbb{K}$ $[0, \varrho]$ if the process LS is an $\mathbb{K}$-local martingale. **If there exists a deflator, then NUPBR holds**.

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Enlargement of filtration results

We define the right-continuous with left limits $\mathbb{F}$-supermartingale

$$Z_t := \mathbb{P}\left(\tau > t \mid \mathcal{F}_t\right).$$

The supermartingale Z coincides with the optional projection of $I_{]\!]0,\tau[\![}$. The decomposition of Z leads to another important martingale that we denote by m, and is given by

$$m := Z + A^{o,\mathbb{F}},$$

where $A^{o,\mathbb{F}}$ is the $\mathbb{F}$-dual optional projection of $A = I_{[\![\tau,\infty[\![}$.

Let $(A_t,\, t \geq 0)$ be an integrable increasing process (not necessarily $\mathbb{F}$-adapted). There exists a unique integrable $\mathbb{F}$-optional increasing process $(A_t^{o,\mathbb{F}}, t \geq 0)$, called the dual optional projection of A such that

$$\mathbb{E}\left(\int_0^\infty Y_s dA_s\right) = \mathbb{E}\left(\int_0^\infty Y_s dA_s^{o,\mathbb{F}}\right)$$

for any positive $\mathbb{F}$-optional process Y.

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In a first step, we restrict our attention to what happens before τ.

Therefore, we do not require any extra hypothesis on τ, since any $\mathbb{F}$ martingale stopped at τ is a $\mathbb{G}$ semi-martingale, as established by Jeulin:

To any $\mathbb{F}$ local martingale M, we associate the $\mathbb{G}$ local martingale $\widehat{M}$

$$\widehat{M}_t^\tau := M_t^\tau - \int_0^{t\wedge\tau} \frac{d\langle M, m\rangle_s^{\mathbb{F}}}{Z_{s-}},$$

and the the $\mathbb{G}$ local martingale $\widetilde{M}$

$$\widetilde{M}_t^\tau := M_t^\tau - \int_0^{t\wedge\tau} \frac{d[M, m]_s^{\mathbb{F}}}{\widetilde{Z}_{s-}},$$

It can be proved that Z and $\widetilde{Z}$ do not vanish on $[0, \tau]$.

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It can be proved that Z and $\widetilde{Z}$ do not vanish on $[0, \tau]$.

Continuous filtrations

If all $\mathbb{F}$ martingales are continuous, there are NA1 before τ

Recall that the bracket of continuous martingales does not depend on the filtration. Let, for $t \leq \tau$,

$$\widehat{m}_t := m_t - \int_0^t \frac{d\langle m, m\rangle_s^{\mathbb{F}}}{Z_s}$$

and define the $\mathbb{G}$ local martingale L as

$$dL_t = L_t d\widetilde{N}_t, \; L_0 = 1, \quad \text{where} \;\; d\widetilde{N}_t = -\frac{1}{Z_t} d\widehat{m}_t.$$

If SL is a local martingale, there are no arbitrages of the first kind. Recall that

$$\widehat{S}_t := S_t - \int_0^t \frac{d\langle S, m\rangle_s^{\mathbb{F}}}{Z_s}$$

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and

$$\widehat{S}_t := S_t - \int_0^t \frac{d\langle S, m\rangle_s^{\mathbb{F}}}{Z_s}$$

we obtain

$$\begin{aligned} d(LS)_t &= L_t dS_t + S_t dL_t + d\langle L, S\rangle_t^{\mathbb{G}} \\ &\stackrel{\text{mart}}{=} L_t \frac{1}{Z_t} d\langle S, m\rangle_t^{\mathbb{F}} + \frac{1}{Z_{t-}} L_t d\langle S, \widehat{m}\rangle_t^{\mathbb{G}} \\ &\stackrel{\text{mart}}{=} L_t \frac{1}{Z_t} \left(d\langle S, m\rangle_t - d\langle S, m\rangle_t\right) = 0 \end{aligned}$$

where $X \stackrel{\text{mart}}{=} Y$ is a notation for $X - Y$ is a local martingale.

Strong arbitrages in the case where $\mathbb{F}$ is the Brownian filtration and τ is an honest time which avoids $\mathbb{F}$ stopping times

A random time τ is honest if τ is equal to an $\mathcal{F}_t$-measurable random variable on $\tau < t$.

Example: Let X be an adapted continuous process and $X^* = \sup X_s, X_t^* = \sup_{s \leq t} X_s$. The random time

$$\tau = \inf\{s : X_s = X^*\}$$

is honest.Indeed, on the set $\{\tau < t\}$, one has $\tau = \inf\{s \leq t : X_s = X_t^*\}$.

If τ is honest, then $Z_\tau = 1$.

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If τ is honest and avoid $\mathbb{F}$ stopping times, then $Z_\tau = 1$.

Arbitrage portfolio

NA fails to hold in the enlarged financial market $\mathcal{M}(\mathbb{G}) = (\Omega, \mathbb{G}, \mathbb{P}; S)$ on the time horizon $[0, \tau]$

The martingale m represents the value of a self-financing portfolio, with initial value 1. Since $m_\tau \geq 1$ and $\mathbb{P}(m_\tau > 1) > 0$, one gets an arbitrage opportunity.

It is possible to prove:

One can never construct arbitrage opportunities in the enlarged financial market $\mathcal{M}(\mathbb{G})$ *strictly before* the honest time τ.

Let ϱ be a $\mathbb{G}$-stopping time with $\varrho < \tau$ $\mathbb{P}$-a.s. Then NFLVR holds in the enlarged financial market $\mathcal{M}(\mathbb{G})$ on the time horizon $[0, \varrho]$.

Arbitrages, General case

The completeness of the $\mathbb{F}$ market seems to be an essential hypothesis to have strong arbitrages:

Let W^1, W^2 be a standard 2-dimensional Brownian motion and

$$dS_t = S_t f(W_t^2) dW_t^1$$

Under regularity assumptions $\mathbb{F}^S = \mathbb{F}^1 \vee \mathbb{F}^2$. Let τ be an $\mathbb{F}^2$ honest time (hence an $\mathbb{F}^S$ honest time). Since W^1 is an $\mathbb{F}^1 \vee \sigma(\tau \wedge \cdot)$ martingale, there are no arbitrages in the enlarged filtration.

Discontinuous case

Poisson case

Let X be a Poisson process, with compensated martingale M and τ a random time.

Let $Z_t = m_t - A_t^{0,p}$ be the optional decomposition of Z and $\widehat{m}$ the $\mathbb{G}$-martingale part of the $\mathbb{G}$ semi-martingale m.

This decomposition is NOT the Doob-Meyer decomposition (see examples below)

In a Poisson setting, from PRP, $dm_t = \psi_t dM_t$ for some predictable process ψ, so that, on $t \leq \tau$,

$$d\widehat{m}_t = dm_t + \frac{1}{Z_{t-}} d\langle m \rangle_t = dm_t + \frac{1}{Z_{t-}} \lambda \psi_t^2 dt$$

We assume that S is an $\mathbb{F}$ martingale.

In a Poisson setting, there are NA1 before τ

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We are looking for a RN density of the form $dL_t = L_{t-}\kappa_t d\widehat{m}_t$ so that $S^\tau L$ is a $\mathbb{G}$ local martingale. Integration by parts formula leads to (on $t \leq \tau$)

$$
\begin{aligned}
d(LS)_t &= L_{t-}dS_t + S_{t-}dL_t + d[L,S]_t \\
&\overset{\text{mart}}{=} L_{t-}S_{t-}\varphi_t \frac{1}{Z_{t-}} d\langle M, m\rangle_t + L_{t-}S_{t-}\kappa_t\varphi_t\psi_t dX_t \\
&\overset{\text{mart}}{=} L_{t-}S_{t-}\varphi_t \frac{1}{Z_{t-}} d\langle M, m\rangle_t + L_{t-}S_{t-}\kappa_t\varphi_t\psi_t \lambda(1 + \frac{1}{Z_{t-}}\psi_t)dt \\
&\overset{\text{mart}}{=} L_{t-}S_{t-}\psi_t\varphi_t\lambda\left(\frac{1}{Z_{t-}} + \kappa_t(1 + \frac{1}{Z_{t-}}\psi_t)\right)dt
\end{aligned}
$$

Therefore, for $\kappa_t = -\frac{1}{Z_{t-}+\psi_t}$, one obtains a deflator. Note that

$$dL_t = L_{t-}\kappa_t d\widehat{m}_t = -L_{t-}\frac{1}{Z_{t-}+\psi_t}\psi_t d\widehat{M}_t$$

is indeed a positive martingale, since $\frac{1}{Z_{t-}+\psi_t}\psi_t < 1$.

Honest times : First Example

Define the time τ as

$$\tau = \sup\{t : \mu t - X_t \leq a\}$$

where $\mu > \lambda$. The Azéma supermartingale associated with the honest time τ is

$$\mathbb{P}(\tau > t | \mathcal{F}_t) = \psi(\mu t - X_t - a) \mathbb{1}_{\{\mu t - X_t \geq a\}} + \mathbb{1}_{\{\mu t - X_t < a\}},$$

where $\psi(x)$ is the ruin probability associated with process $\mu t - X_t$ and starting point $x > 0$, i.e., $\psi(x) = \mathbb{P}(T^x < \infty)$ with $T^x = \inf\{t : x + \mu t - X_t < 0\}$.

Define $\vartheta_1 = \inf\{t > 0 : \mu t - X_t = a\}$ and then, for each $n > 1$, $\vartheta_n = \inf\{t > \vartheta_{n-1} : \mu t - X_t = a\}$.

The dual optional projection $A^{o,\mathbb{F}}$ of the process $1\!\!1_{[\tau,\infty)}$ equals

$$A^{o,\mathbb{F}} = \frac{\theta}{1+\theta} \sum_n 1\!\!1_{[\vartheta_n,\infty)}$$

where $\theta = \frac{\mu}{\lambda} - 1$ and

$$m_t = \frac{\theta}{1+\theta} \sum_n 1\!\!1_{(t \geq \vartheta_n)} + \psi(\mu t - X_t - a) 1\!\!1_{\{\mu t - X_t \geq a\}} + 1\!\!1_{\{\mu t - X_t < a\}}$$

Strong arbitrages:

Note that the process $A^{o,\mathbb{F}}$ is flat after τ and that, on the set $\tau = \vartheta_n$, one has $A^{o,\mathbb{F}}_\tau = \frac{\theta}{1+\theta} n$. The martingale m takes the value 1 at time 0 and

$$m_\tau = Z_\tau + \frac{\theta}{1+\theta} n = \frac{1}{1+\theta} + \frac{\theta}{1+\theta} n = \frac{1}{1+\theta}(1 + \theta n)$$

therefore $m_\tau \geq 1$ and $\mathbb{P}(m_\tau > 1) > 0$. Since the market is complete, this martingale is the value of a portfolio. Note that $m_t = Z_t + A^{o,\mathbb{F}}_t \geq Z_t > 0$, hence the strategy is admissible.

Honest times : Second Example

Let

$$dS_t = S_{t-}\varphi dM_t,\ S_0 = 1$$

or

$$S_t = \exp(-\lambda\varphi t + \ln(1+\varphi)N_t)\,.$$

The process $S_t^* = \sup_{s\le t} S_s$ is continuous if $\varphi < 0$.

Define the random time τ as

$$\tau = \sup\{t : S_t = S_t^*\}.$$

Let us note that τ is well defined and that if $\varphi > 0$ $S_\tau < S_\tau^* = \sup_t S_t$

if $-1 < \varphi < 0$, $S_\tau = S_\tau^* = \sup_t S_t$.

The time τ does not avoid $\mathbb{F}$-stopping times, and is not an $\mathbb{F}$ stopping time. There are arbitrages if $\varphi > 0$, there are no arbitrages if $\varphi < 0$.

The Azéma supermartingale associated with the honest time τ is

$$\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\sup_{s\in(t,\infty]} S_s \geq \sup_{s\in[0,t]} S_s|\mathcal{F}_t) = \mathbb{P}(\sup_{s\in[0,\infty]} \widehat{S}_s \geq \frac{S_t^*}{S_t}|\mathcal{F}_t) = \psi(\frac{S_t^*}{S_t}),$$

with $\widehat{S}$ an independent copy of S and $\psi(x) = \mathbb{P}(S^* \geq x)$.

If $\varphi > 0$, $S_\tau^* = S_\tau$, hence $Z_\tau = 1$. It follows that $m_\tau > 1$, hence m is the value of a self financing strategy associated with an arbitrage.

If $\varphi < 0$, S^* is continuous and

$$dZ_t \quad = \quad \left(\psi(\frac{S_t^*}{S_{t-}(1+\varphi)}) - \psi(\frac{S_t^*}{S_{t-}})\right) dN_t + \psi'(\frac{S_t^*}{S_{t-}}) \left(\varphi\lambda \frac{S_t^*}{S_{t-}} dt + \frac{1}{S_t} dS_t^*\right)$$

Then, $m_t = 1 + \int_0^t \Delta_s dM_s$ and, on $t \leq \tau$

$$\hat{M}_t = M_t - \int_0^t \frac{\Delta_s}{Z_s} \lambda ds = N_t - \int_0^t \lambda(1 + \frac{\Delta_s}{Z_s}) ds$$

where $\Delta_s = \psi(\frac{S_s^*}{S_{s-}(1+\varphi)}) - \psi(\frac{S_s^*}{S_{s-}})$.

The quantity $(1 + \frac{\Delta_s}{Z_s})$ is positive: indeed

$$(1 + \frac{\Psi_s}{Z_s}) = \frac{1}{Z_s}(Z_s + \Psi_s) = \frac{1}{Z_s}\left(\psi(x) + \psi(\frac{x}{1+\varphi}) - \psi(x)\right)|_{x = S_s^*/S_s}$$

Hence, there exists a change of probability so that M is a $\mathbb{G}$-martingale.

The Case of Quasi-Left Continuous Processes

This subsection focuses on processes that do not jump on predictable stopping times (i.e., quasi-left continuous processes). We prove that NA1 is preserved under random horizon for these processes under some additional assumptions.

We assume that S and m are quasi-left continuous processes. We also assume that Z and Z_- are strictly positive. In all this section, the processes are considered on the time interval $]\!]0, \tau[\![$.

Consider the $\mathbb{G}$-local martingale $\widehat{m}$ and the process $K := (\widetilde{Z})^{-1}$ where $\widetilde{Z}_t = \mathbb{P}(\tau \geq t | \mathcal{F}_t)$. It is known that $Z_- + \Delta m = \widetilde{Z}$.

Optional Integral

Let N be a local martingale and H an adapted process.

(a) **The compensated stochastic integral** $M = H \odot N$ is the unique $\mathbb{K}$-local martingale such that, for any $\mathbb{K}$-local martingale Y,

$$\mathbb{E}\left([M,Y]_\infty\right) = \mathbb{E}\left(\int_0^\infty H_s d[N,Y]_s\right).$$

(b) The process $[M,Y] - H \centerdot [N,Y]$ is an $\mathbb{K}$-local martingale.

The compensated stochastic integral of H with respect to N is the unique local martingale, M, such that

$$M^c = {}^{p,\mathbb{K}}H \centerdot N^c \quad \text{and} \quad \Delta M = H\Delta N - {}^{p,\mathbb{K}}(H\Delta N)$$

where ${}^{p,\mathbb{K}}X$ is the predictable projection of the process X.

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The process $\mathcal{E}\left(\widetilde{N}\right)S^{\tau}$ is a positive $\mathbb{G}$-local martingale, where the process $\widetilde{N} := -K \odot \widehat{m}$ is a $\mathbb{G}$-local martingale.

We prove that $\mathcal{E}(\widetilde{N}) > 0$, or equivalently $1 + \Delta\widetilde{N} > 0$. From the definition of optional integrals

$$1 + \Delta\widetilde{N} \quad = \quad 1 - \frac{\Delta\widehat{m}}{\widetilde{Z}} + {}^{p,\mathbb{G}}\left(\frac{\Delta\widehat{m}}{\widetilde{Z}}\right)$$

Using the fact that $\Delta\widehat{m} = \Delta m$ and that, $K = \widetilde{Z}^{-1} = (Z_- + \Delta m)^{-1}$, we obtain

$$1 + \Delta\widetilde{N} \quad = \quad 1 - \frac{\Delta m}{\widetilde{Z}} + {}^{p,\mathbb{G}}\left(\frac{\Delta m}{\widetilde{Z}}\right) = \frac{Z_-}{\widetilde{Z}} > 0$$

Indeed, for any predictable stopping time T we have

$${}^{p,\mathbb{G}}\left(\frac{\Delta m}{\widetilde{Z}}\right)_T \mathbb{1}_{(T<\infty)} = \mathbb{E}(\frac{\Delta m_T}{\widetilde{Z}_T}\mathbb{1}_{(T<\infty)}|\mathcal{F}_{T-}) = 0$$

Assuming that S is quasi-left continuous

$$\begin{aligned}
1\!\!1_{]0,\tau]} \centerdot [\widehat{m}, \widehat{S}] &= 1\!\!1_{]0,\tau]} \centerdot [m, S] - \frac{1}{Z_-} 1\!\!1_{]0,\tau]} \centerdot [\langle m \rangle^{\mathbb{F}}, S] - \frac{1}{Z_-} 1\!\!1_{]0,\tau]} \centerdot [\widehat{m}, \langle m \rangle^{\mathbb{F}}] \\
&= 1\!\!1_{]0,\tau]} \centerdot [m, S]
\end{aligned}$$

since $\langle m \rangle^{\mathbb{F}}$ and S have no common jumps and $\langle m \rangle^{\mathbb{F}}$ is continuous. It follows that

$$\begin{aligned}
[\widetilde{N}, S] &= [\widetilde{N}, \widehat{S}] + [\widetilde{N}, \frac{1}{Z_-} 1\!\!1_{]0,\tau]} \centerdot [\langle m, S \rangle^{\mathbb{F}}] + [-\frac{1}{\widetilde{Z}} 1\!\!1_{]0,\tau]} \odot \widehat{m}, \widehat{S}] \\
&\quad + \frac{1}{Z_-} 1\!\!1_{]0,\tau]} \Delta \langle m, S \rangle^{\mathbb{F}} \centerdot \widetilde{N} \\
&= \frac{1}{Z_-} 1\!\!1_{]0,\tau]} \centerdot [\widehat{m}, \widehat{S}] = \frac{1}{Z_-} 1\!\!1_{]0,\tau]} \centerdot [m, S]
\end{aligned}$$

General case

Let τ be a random time. Then, the following assertions are equivalent:

(i) The thin set $\{\widetilde{Z} = 0 \cap Z_- > 0\}$ is evanescent.

(ii) For any process S satisfying NUPBR($\mathbb{F}$), S^τ satisfies NUPBR($\mathbb{G}$).

After τ, honest times

We have to impose condition on τ so that, after τ, $\mathbb{F}$ martingales are $\mathbb{G}$ semi-martingales.

We restrict our attention to the case of honest times. We recall that we use the additive decomposition of Z of the form

$$Z_t = m_t - A_t^{o,p}$$

Then, any $\mathbb{F}$ martingale X is a $\mathbb{G}$ semimartingale with decomposition

$$X_t = \widetilde{X}_t + \int_0^{t\wedge\tau} \frac{d\langle X, m\rangle_s}{Z_{s-}} - \int_\tau^{\tau\vee t} \frac{d\langle X, m\rangle_s^{\mathbb{F}}}{1 - Z_{s-}}\,,$$

where $\widetilde{X}$ such that is a $\mathbb{G}$-local martingale.

Brownian case

We assume that τ avoids $\mathbb{F}$ stopping times. Then $Z_\tau = 1$.

The process $m - m_\tau$ yields to a strong arbitrage.

The r.v. m_τ yields to an arbitrage of the first kind.

Indeed, for $t > \tau$, $m_t = Z_t + A^o_t = Z_t + A^o_\tau < 1 + m_\tau - 1 = m_\tau$ and $m_t = m_\tau + \int_\tau^t \varphi_s dS_s$

Quasi continuous case

Assume that m and S are qcl

Let $\widehat{m} = 1\!\!1_{[\tau,\infty[} \cdot m + \frac{1}{1-Z_-} \cdot \langle m \rangle^{\mathbb{F}}$ and $\tilde{Z}$ the supermartingale $\tilde{Z}_t = \mathbb{P}(\tau \geq t|\mathcal{F}_t)$

Define $\widetilde{N} = \frac{1}{1-\tilde{Z}} \odot \widehat{m}$

We see that $\mathcal{E}(\widetilde{N})$ is positive $\mathbb{G}$-local martingale

We prove, using that $\langle m, S \rangle^{\mathbb{F}}$ is continuous, that for a $\mathbb{G}$-martingale $\widetilde{N}$ for every $\mathbb{F}$-martingale S we have

$$\frac{1}{1-\widetilde{Z}} 1\!\!1_{]\tau,\infty[} \cdot [m, S] = [\widetilde{N}, S]$$

or equivalently $\mathcal{E}(\widetilde{N})(S - S^\tau)$ is $\mathbb{G}$-local martingale for each $\mathbb{F}$-martingale S.

General result

We recall that a random set A is called evanescent if the set $\{\omega, \exists t(\omega,t) \in A]$ is $\mathbb{P}$ null A random time τ is called a thin random time if its graph is contained in a thin set, i.e., if there exists a sequence of $\mathbb{F}$-stopping times $(\vartheta_n)_{n=1}^{\infty}$ with disjoint graphs such that $[\![\tau 1]\!] \subset \bigcup_n [\![\vartheta_n]\!]$.

Let τ be a random time satisfying $Z_\tau < 1$. Then, the following assertions are equivalent:
(i) The thin set $\{\widetilde{Z} = 0 \cap Z_- > 0\}$ is evanescent.
(ii) For any process S such that $S - S^\tau$ satisfies NUPBR($\mathbb{F}$), $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).

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Thank you for your attention