Financial market models defined by a random preference relation. Essential supremum and maximum of a family of random variables with respect to a random preference relation. Applications.

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• Y. Kabanov and E.Lépinette. Essential supremum with respect to a random partial order. To appear in *Journal of Mathematical Economics*.

• Y. Kabanov and E.Lépinette. Essential supremum and essential maximum with respect to random preference relations. To appear in *Journal of Mathematical Economics*.

Modeling a discrete-time financial market with a random preference relation (or preorder)

☞ In the real world, a portfolio is expressed in physical units, i.e. the number of risky assets an agent holds. Even worse, these quantities are integer-valued (except the cash account but we can change the monetary unit).

☞ In practice, there are various kinds of transaction costs generated by taxes, bid-ask spread, etc.. See for instance the models of Schachermayer and Kabanov including proportional transaction costs.

Modeling a discrete-time financial market with a random preference relation (or preorder)

☞ The concepts of liquidation value and solvency are fundamental.

☞ At time t, can we rebalance a self-financing portfolio position $x_{t-1} \in \mathbf{R}^d$ into $x_t \in \mathbf{R}^d$?
To do so, split the portfolio into two parts : $x_{t-1} = x_t + (x_{t-1} - x_t)$ and liquidate (if possible, i.e. without any debt) the position $x_{t-1} - x_t$.

Modeling a discrete-time financial market with a random preference relation (or preorder)

☞ We may introduce a stochastic liquidation function L_t so that we can rebalance a self-financing portfolio position $x_{t-1} = x_t + (x_{t-1} - x_t) \in \mathbf{R}^d$ into $x_t \in \mathbf{R}^d$ iff $L_t(x_{t-1} - x_t) \geq 0$.

☞ More generally, if we consider the random set of solvable positions G_t, then we require that $x_{t-1} - x_t \in G_t$.
In the Kabanov model, G_t is the so called (random) solvency cone.

Modeling a discrete-time financial market with a random preference relation (or preorder)

Let us consider a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\cdots,T}, P)$. Let $(P_t(x))_{t=0,\cdots,T}$ be a Caratheodory function on $\Omega \times \{0, \cdots, T\} \times \mathbf{R}^d$, i.e. satisfying :

- (a) : For each ω P-a.s., and every $t = 0, \cdots, T$, $P_t(\omega, \cdot)$ is continuous on $\mathbf{R}^d$.
- (b) : For each $(t, x) \in \{0, \cdots, T\} \times \mathbf{R}^d$, $P_t(\cdot, x) \in L^0(\mathbf{R}, \mathcal{F}_t)$.
- (c) : $P_t(0) \geq 0$ a.s. for all $t = 0, \cdots, T$,
- (d) : For all $t = 0, \cdots, T$ the property $(P_t(x) \geq 0$ and $P_t(y) \geq 0)$ for some $x, y \in \mathbf{R}^d$ implies $P_t(x + y) \geq 0$ holds a.s.

Modeling a discrete-time financial market with a random preference relation (or preorder)

☞ Example, $P_t(x) = -d(x, G_t)$ where the graph of G_t is $\mathcal{F}_t \times \mathcal{B}(\mathbf{R}^d)$-measurable such that $0 \in G_t$, $G_t + G_t \subseteq G_t$ and G_t is a.s. closed.

Definition

A portfolio process $(V_t)_{t=0,\cdots,T}$ is an $(\mathcal{F}_t)_{t=0,\cdots,T}$-adapted process such that

$$P_t(V_{t-1} - V_t) \geq 0, \quad \forall t = 0, \cdots, T \quad a.s. \tag{0.1}$$

☞ For each t, $\gamma_1 \succeq^t \gamma_2$ if $P_t(\gamma_1 - \gamma_2) \geq 0$ is a preorder on $L^0(\mathbf{R}^d, \mathcal{F})$.

Modeling a discrete-time financial market with a random preference relation (or preorder)

The graph

$$\mathrm{GR}(t) := \{(\gamma_1, \gamma_2) \in L^0(\mathbf{R}^d, \mathcal{F}) \times L^0(\mathbf{R}^d, \mathcal{F}) : \ \gamma_2 \succeq^t \gamma_1\}$$

is closed in $L^0(\mathbf{R}^d, \mathcal{F}) \times L^0(\mathbf{R}^d, \mathcal{F})$ since the function P satisfies Condition (a). It follows that the preorder $\succeq^t$ admits both a lower and upper semi-continuous multi-utility representation (❄).
☞ We may also think for each ω : $x \succeq^{t,\omega} y$ iff $P_t(\omega, x - y) \geq 0$.
By Evren and Ok, as $\mathbf{R}^d$ is locally compact and σ-compact, the random preorder $\succeq^{t,\omega}$ has a **countable** continuous multi-utility representation, i.e. a family $\mathcal{U} = \mathcal{U}(\omega)$ of functions (u_i) such that

$$x \succeq^{t,\omega} y \quad \text{iff} \quad u_i(x) \geq u_i(y), \qquad \text{for all } i.$$

❄ Evren O., Ok E.A. On the multi-utility representation of preference relations. *Journal of mathematical economics*, 14 (2011), 4-5, 554-563.

The Kabanov model with proportional transaction costs

• The random set G_t is a polyhedral closed convex cone containing $\mathbf{R}^d_+$ corresponding to the portfolios a time t whose positions can be changed, paying transaction costs, into positives ones.

☞ $x \succeq^{t,\omega} y$ means $x - y \in G_t(\omega)$ we also denote by $x \geq_{G_t} y$.
☞ There exists a countable multi-utility representation of the random preorder $\succeq^{t,\omega}$, precisely a family of random linear mappings $u_i(t,x) = \xi_i(t,\omega)x$.

• If two positions $x, y \in \mathbf{R}^d$ are such that $x \geq_{G_t} y$, i.e. $x - y \in G_t$, that means that y is cheaper than x.

A two-dimensional model with bid-ask spread and fixed transaction costs

- The first position is a Cash account with $S^1 = 1$ on $[0, T]$ and the second one is risky and modeled by $S = S^2$.
- We suppose that there is a bid-ask spread $[S(1-\epsilon); S(1+\epsilon)]$.
- There are only transaction costs for the second position towards the first one, precisely a fixed cost for each transaction we denote by c.
- Besides, when $y \geq 0$, we suppose that the agent is rational enough not to deliberately sell the stock when the bid-price is too low to compensate for the fixed cost, i.e. $S_t(1-\epsilon)y - c_t \leq 0$. Let us characterize $(x, y) \in G_t$.

A two-dimensional model with bid-ask spread and fixed transaction costs

- If $y \geq 0$, this means that $x + S_t(1-\epsilon_t)y - c_t \geq 0$ or $x \geq 0$.
- If $y < 0$, this means that $x + S_t(1+\epsilon_t)y - c_t \geq 0$.

i.e.

$$
\begin{aligned}
&z = (x, y) \succeq^t z' = (x', y') \\
&\Leftrightarrow \max(x - x' + S_t(1-\epsilon_t)(y - y') - c_t, x - x') \geq 0, \\
&\text{and } (x - x' - c_t)^+ + S_t(1+\epsilon_t)(y - y') \geq 0
\end{aligned}
$$

A two-dimensional model with bid-ask spread and fixed transaction costs

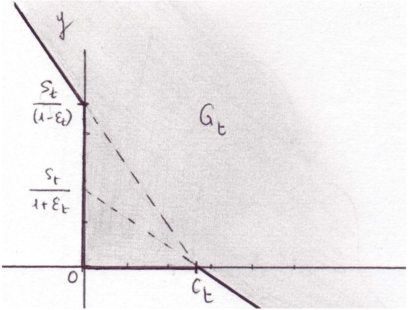


Figure: The grey-coloured domain corresponds to the set G_t of solvent points.

Super-hedging portfolio processes in a discrete-time financial market model defined by a preorder

• A portfolio V starting from $V_{0-} = 0$ is an $\mathbf{R}^d$-valued process satisfying the dynamics $V_{t-1} \succeq^t V_t$ for all $t = 0, \dots, T$.

• It super replicates the European claim $h_T \in L^0(\mathbf{R}^d, \mathcal{F}_T)$ (resp. the American claim $(h_t)_{t=0,\cdots,T}$) if $V_T \succeq^T h_T$ (resp. $V_t \succeq^t h_t$ for all t).

Super-hedging portfolio processes in a discrete-time general financial market model

☞ We seek for a sub class $\mathcal{V}_h^{min}$ of the super-replicating portfolio processes $\mathcal{V}_h$ of a given payoff h we call minimum in the following sense :

- if $V \in \mathcal{V}_h$, there exists $\hat{V} \in \mathcal{V}_h^{min}$ such that $V_t \succeq^t \hat{V}_t$ for all t,
- if $\hat{V} \in \mathcal{V}_h^{min}$ and $\hat{V}_t \succeq^t V_t \; \forall t, V \in \mathcal{V}_h$ then $\hat{V}_t \sim^t V_t$ for all t.

☞ Without transaction costs, the minimal super replicating portfolio price of an European claim (resp. American claim) is unique and defined using the concept of essential supremum of a family of random variables.

How to generalize the concept of Essential Supremum of a collection of real-valued random variables to a family of vector-valued random variables ?

Definition

Let $(\xi_i)_{i \in I}$ be a family of real-valued random variables. There exists a unique random variable $\eta \in (-\infty, \infty]$ satisfying the following properties :

(1) $\eta \geq \xi_i$, $\forall i \in I$.

(2) If $\eta' \geq \xi_i$, $\forall i \in I$, then $\eta' \geq \eta$.

How to define a measurable version of the Pareto frontier ?

Definition

Let X be a set endowed with a family of utility functions $(u_i)_{i\in I}$. An element $x \in X$ dominates (resp. strictly dominates) $y \in X$ if $u_i(x) \geq u_i(y)$ for all $i \in I$ (resp. $u_i(x) \geq u_i(y)$ for all $i \in I$ and $u_j(x) > u_j(y)$ for some $j \in I$).

Definition (Pareto frontier)

The Pareto frontier of X is the subset of X containing the efficient points of X, i.e. the points of X which are not strictly dominated.

Basic notations

• Define an order interval $[x, y] := \{z \in X : \ x \succeq z \succeq y\}$ and extend naturally the notation by putting

$$]-\infty, x] := \{z \in X : \ x \succeq z\}, \qquad [x, \infty[:= \{z \in X : \ z \succeq x\}.$$

• The notation $\Gamma_1 \succeq \Gamma_2$ where Γ_1, Γ_2 are subsets means that $x_1 \succeq x_2$ for all $x_1 \in \Gamma_1$ and $x_2 \in \Gamma_2$;
$[\Gamma_1, \infty[:= \cap_{x_1 \in \Gamma_1} \{z \in X : \ z \succeq x_1\}$ etc.

Essential supremum in $L^0(X)$

The Model

• Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{F}$.

• We consider in the space $L^0(X)$ of X-valued random variables a random preorder defined by a countable family $\mathcal{U} = \{u_j : \ j = 1, 2, ...\}$ of functions $u_j : \Omega \times X \to \mathbf{R}$ with the following properties :

(i) $u_j(., x) \in L^0(\mathbf{R}, \mathcal{F})$ for every $x \in X$;

(ii) $u_j(\omega, .)$ is continuous for almost all $\omega \in \Omega$.

• If $\gamma_1, \gamma_2 \in L^0(X, \mathcal{F})$ the relation $\gamma_2 \succeq \gamma_1$ means that $u_j(\gamma_2) \geq u_j(\gamma_1)$ (a.s.) for $j = 1, 2, ...$

Definition

Let Γ *be a subset of* $L^0(X, \mathcal{F})$*. We denote by* $\mathcal{H}\text{-}\mathrm{Esssup}\,\Gamma$ *the maximal subset* $\hat{\Gamma}$ *of* $L^0(X, \mathcal{H})$ *such that the following conditions hold :*

(a) $\hat{\Gamma} \succeq \Gamma$ *;*

(b) if $\gamma \in L^0(X, \mathcal{H})$ *and* $\gamma \succeq \Gamma$*, then there is* $\hat{\gamma} \in \hat{\Gamma}$ *such that* $\gamma \succeq \hat{\gamma}$ *;*

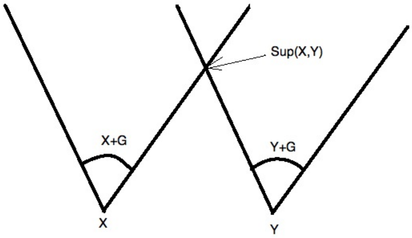
(c) if $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$*, then* $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ *implies* $\hat{\gamma}_1 \sim \hat{\gamma}_2$*.*

Supremum with respect to a cone $G \subset \mathbf{R}^d$

Theorem

Let $\succeq$ be the partial order generated by a closed proper convex cone $G \subseteq \mathbf{R}^d$. If $\Gamma \subseteq \mathbf{R}^d$ is such that $\bar{x} \succeq \Gamma$ (i.e. $\bar{x} - \Gamma \subseteq G$) for some $\bar{x} \in \mathbf{R}^d$, then $\operatorname{Sup} \Gamma \neq \emptyset$.

☞ The Supremum of two points X and Y in $\mathbf{R}^2$ with respect to a cone G :



Essential supremum in $L^0(\mathbf{R}^d)$: existence

Theorem

Let $\succeq$ be a partial order in $L^0(\mathbf{R}^d)$ represented by a countable family of random functions satisfying (i), (ii) and such that all order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$, $\gamma_2 \succeq \gamma_1$, are compacts a.s.. If the subset $\Gamma \subseteq L^0(\mathbf{R}^d, \mathcal{F})$ is such that $\bar{\gamma} \succeq \Gamma$ for some $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{H})$, then $\mathcal{H}\text{-}\mathrm{Esssup}\,\Gamma \neq \emptyset$.

Concept of *Essential Maximum*

Definition

Let $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{F}$. We say that a non-empty subset Γ of $L^0(X, \mathcal{F})$ is $\mathcal{H}$-decomposable if for any finite $\mathcal{H}$-measurable partition $(A_i)_{i=1}^n$ of Ω and all sequence $(\gamma_i)_{i=1}^n$ of Γ, $\sum_{i=1}^n 1_{A_i}\gamma_i \in \Gamma$.

Definition

Let Γ be a non-empty subset of $L^0(X, \mathcal{F})$ and $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{F}$. We denote by $\Gamma_{\mathcal{H}}$ the $\mathcal{H}$-decomposable envelop of Γ, i.e. the smallest subset of $L^0(X, \mathcal{F})$ which is $\mathcal{H}$-decomposable and contains Γ.

Concept of *Essential Maximum*

Definition

Let Γ *be a non-empty subset of* $L^0(X, \mathcal{F})$*. We denote by* $\mathrm{Essmax}_1\Gamma$ *the largest subset* $\hat{\Gamma} \subseteq \overline{\Gamma_{\mathcal{H}}}$ *such that the following conditions hold :*

(i) *if* $\gamma \in \overline{\Gamma_{\mathcal{H}}}$*, then there is* $\hat{\gamma} \in \hat{\Gamma}$ *such that* $\hat{\gamma} \succeq \gamma$ *;*

(ii) *if* $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$*, then* $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ *implies* $\hat{\gamma}_1 = \hat{\gamma}_2$*.*

Concept of *Essential Maximum*

Definition

Let Γ *be a non-empty subset of* $L^0(\mathbf{R}^d, \mathcal{F})$. *We put*

$$\mathrm{Essmax}\,\Gamma = \{\gamma \in \overline{\Gamma_{\mathcal{H}}} :\ \overline{\Gamma_{\mathcal{H}}} \cap [\gamma, \infty[= [\gamma]\}.$$

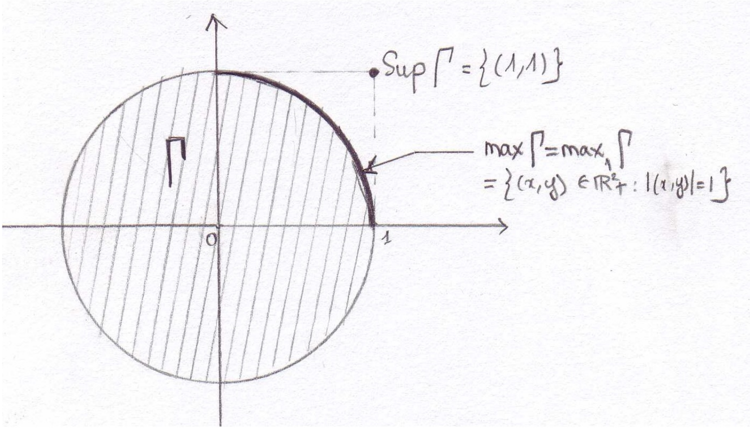
Essential Maximum : existence

Proposition

Let $\succeq$ be a partial order in $L^0(\mathbf{R}^d, \mathcal{F})$ represented by a countable family of functions satiffying (i), (ii) and such that all order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$, $\gamma_2 \succeq \gamma_1$, are compacts a.s. Let Γ be a non-empty subset of $L^0(\mathbf{R}^d, \mathcal{H})$. Suppose that there exists $\bar{\gamma} \in L^0(\mathbf{R}^d, \mathcal{F})$ such that $\bar{\gamma} \succeq \Gamma$. Then $\mathrm{Essmax}_1\Gamma$ and $\mathrm{Essmax}\Gamma$ are non-empty sets and $\mathrm{Essmax}_1\Gamma = \mathrm{Essmax}\,\Gamma$.

Example in the deterministic case

• Let $X = \mathbf{R}^2$, the partial order is generated by the cone $\mathbf{R}^2_+$. Let $\Gamma = \{x : \ |x| \leq 1\}$. Then $\mathrm{Sup}\,\Gamma = (1,1)$ while $\mathrm{Max}\,\Gamma = \{x : \ |x| = 1\} \cap \mathbf{R}^2_+$.



Minimal super-hedging prices in a discrete-time financial market models of Kabanov

The usual discrete-time models with proportional transaction costs can be described in an abstract setting as follows :

• Let $(\Omega, F, \mathcal{F} = (\mathcal{F}_t)_{t=1,\ldots,T}, P)$ be a filtered probability space.

• Let $(G_t)_{t=0,\ldots,T}$ be polyhedral random cones in $\mathbf{R}^d$ s.t. the graph $\Delta_t := \{(\omega, x) : x \in G_t(\omega)\}$ is $\mathcal{F}_t \times \mathcal{B}(\mathbf{R}^d)$-measurable for each t.

Minimal super-hedging prices in a discrete-time financial market model of Kabanov

A portfolio V starting from $V_{0-} = 0$ is an $\mathbf{R}^d$-valued process satisfying the dynamics $\Delta V_t := V_t - V_{t-1} \in -G_t$ $(V_{t-1} \succeq^t V_t)$ for all $t = 0, \ldots, T$.

Proposition

Suppose that $L^0(G_{t+1},\mathcal{F}_t) \subseteq L^0(G_t,\mathcal{F}_t)$, $t \le T-1$ *and suppose there exists a least one* $V \in \mathcal{V}$ *such that* $V_T \ge_{G_T} h_T$. *Then* $\mathcal{V}_{min} \neq \emptyset$ *and* $\mathcal{V}_{min}$ *coincides with the set of solutions of backward inclusions*

$$V_t \in (\mathcal{F}_t, G_{t+1})\text{-Esssup}\,\{V_{t+1}\}, \quad t \le T-1, \quad V_T = h_T. \quad (0.2)$$

Moreover, any $W \in \mathcal{V}$ *with* $W_T \succeq Y_T$ *is such that* $W \succeq_G V$ *for some* $V \in \mathcal{V}_{min}$.

Proposition

Suppose there exists a process $V \in \mathcal{V}$ such that $V \succeq_G h$. Then the set $\mathcal{V}_{min}$ is non-empty and coincides with the set of solutions of backward inclusions

$$V_t \in (\mathcal{F}_t, G_t)\text{-}\mathrm{Essmin}_1 L^0((h_t + G_t) \cap (V_{t+1} + G_{t+1}), \mathcal{F}_t),$$
$$t \leq T-1, \quad V_T = h_T.$$

Thank you for your attention !

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